FITTING AND FORMATION THEORY

IN

LOCALLY FINITE GROUPS.

by

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A dissertation submitted for the degree of

Doctor of Philosophy

in the University of Warwick.

1973.

ERRATA.

PAGE	LINE	COMMENT
5	28	Delete 'and is'.
8	24	Should read '... be such that ...' .
11	25	Delete '$L\mathfrak{N}$' and insert '$L\mathfrak{N} \cap \mathfrak{W}$'.
18	11	We require 1.1.6, for $\prod_{p\in\pi} S_p$ to make any sense, the proof of which is independent of 1.1.3.
23	6	Replace '$S_{p'}$' by '$S_{p'}H/H$' .
50	6	Replace 'H' by 'E'.
57	15	Delete 'By Theorem 1.3.7' and insert 'Now..' and insert after '..such that' the statement ' $\underset{\sim}{T}^{\alpha} = \underset{\sim}{S} \cap E_{\sigma-1}$ whence ... ' in line 9.
57	18	Delete the word 'Hence'.
70	12	Should read '...and let $x^* \in H^* - C^*$.'.
72	29	Insert '...and DF = G, by Theorem 1.3.7.'.
73	3	Delete 'SD' and insert '$\grave{P}\mathfrak{A}$'.
88	19	This is not quite correct - the locally inner automorphisms involved in $\mathfrak{U}$-groups are only those which are effected locally by conjugation by elements of the locally nilpotent residual of the group concerned.
90	20	Should read 'Let $T^*_{p'}$ be a Sylow p'-subgroup of G containing $T_{p'}$. Then $\bigcap_{q\neq p} T^*_{q'}$ is a p-group, and since T_p is a Sylow p-subgroup of G, $T_p = \bigcap_{q\neq p} T^*_{q'}$.
91	6	For $p \neq 2$, choose X_p to be a conjugate of X by an element of C_p.

PAGE	LINE	COMMENT
101	1	$\mathfrak{F}$ is an arbitrary Fitting class of $\mathcal{V}$-groups.
108	20	$K_{L\mathcal{G}}$ is the maximal normal $L\mathcal{G}$-subgroup of K.
120	10	The point is that $S \cap E \nleq R$ since otherwise $E \leq RY = H$ and G/H is a 3-group.
124	29	A correct proof of (A3) (b) is ..... $N = M^*P^*$, where M^* is a P-submodule of M and $P^* = P \cap N$. Then $M = M^* \times M_o$ for some P-submodule M_o of M. Clearly we may now choose the $m_n \in M_o$ such that $(F_nN/N)^{\beta} = F_n^{m_n}N/N$. Let $a_i = m_i m_{i+1}^{-1} \in M_o$. Then $[a_i , F_i] \leq M^*P^* \cap M_o = 1$. The proof now continues as before from line 9 onwards.

CONTENTS.

INTRODUCTION.

ABSTRACT.

The theory of saturated formations introduced by Gaschütz 7 in 1963 is now an integral part of the study of finite soluble groups. Extensions of this theory have since been obtained by Stonehewer 18, for the class of periodic locally soluble groups with a normal locally nilpotent subgroup of finite index, and Tomkinson 21, for the class of periodic locally soluble FC -groups. In the latter case of course conjugacy of the various types of subgroups concerned is replaced by local conjugacy. Wehrfritz 23 also developed a theory of basis normalizers and Carter subgroups for the class of all homomorphic images of periodic soluble linear groups. Much of this work was unified in 1971 in a paper by Gardiner, Hartley and Tomkinson 6. They introduced a class $\mathfrak{U}$ of periodic locally soluble groups and showed that it is possible to obtain a theory of saturated formations in any subclass of $\mathfrak{U}$ which is closed under taking subgroups and homomorphic images. Their work covers all the previous theories except that for periodic locally soluble FC - groups.

It is our aim to show that, with a 'good' Sylow structure and a 'well behaved' group of automorphisms which permutes the Sylow structure transitively, a theory of saturated formations can always be constructed. The approach which we shall discuss gives a theory which covers

all the previous theories including that for periodic locally soluble FC -groups.

This thesis is divided into two main parts and is organised as follows. In the next section we introduce our notation and terminology.

In part one, which forms the bulk of the thesis, we shall define axiomatically what we mean by 'good' and 'well behaved', and in so doing introduce a class $\mathcal{W}$ of periodic locally soluble groups. We shall consider a fixed, but arbitrary, QS -closed subclass $\mathcal{K}$ of $\mathcal{W}$ and a saturated $\mathcal{K}$ -formation $\mathfrak{F}$ satisfying certain conditions, and show that any $\mathcal{K}$ -group G possesses a unique A(G) -transitive class of $\mathfrak{F}$ -projectors, where A(G) is a 'well behaved' group of automorphisms of G. In the final section of part one we extend the work of Chambers [4] to $\mathcal{K}_A$ -groups, and in particular we characterize the $\mathfrak{F}$ -normalizers of $\mathcal{K}_A$ -groups by the covering/avoiding property.

In part two, we introduce a class $\mathcal{U}$ of periodic locally soluble groups, where the automorphisms involved are the locally inner ones; this class properly contains the class of periodic locally soluble FC -groups. We define what we mean by a Fitting class $\mathfrak{F}$ of $\mathcal{U}$ -groups, and show that any $\mathcal{U}$ -group possesses a unique local conjugacy class of $\mathfrak{F}$ -injectors. This extends the work of Tomkinson [22], and we follow his techniques when proving the above. The final sections of part two concern normal Fitting classes of $\mathcal{U}$ -groups, where we extend the work of Blessenohl and Gaschütz [2] and Lausch [15]. In particular we show that every non - trivial normal Fitting class of $\mathcal{U}$-groups admits one and only one normal Fitting pair (up to isomorphic Fitting pairs).

The work in this thesis, apart from the results attributed to others, is to the best of my knowledge original.

I am grateful to the Science Research Council for its financial support during the period 1970 - 1973 when this work was carried out. I would also like to thank Dr. Brian Hartley and Dr. Ian Stewart for many interesting and helpful conversations. In particular, I would like to especially thank my supervisor Dr. Stewart Stonehewer who was a constant source of ideas and encouragement and without whom this thesis would never have achieved its final form.

NOTATION AND TERMINOLOGY.

In this section we introduce the notation and terminology which we shall use throughout this thesis.

We shall use capital Roman letters to denote groups and small Roman letters for elements of groups. As usual $\leq$, $<$ respectively denote 'is a subgroup of', 'is a proper subgroup of'. If X is a subset of a group G, then $< X >$ denotes the subgroup of G generated by the elements of X.

For any set of elements S of a group G, the normalizer of S in G is denoted by $N_G(S)$, and the centralizer of S in G is denoted by $C_G(S)$. The smallest normal subgroup of G containing S is called the normal closure of S in G, and is denoted by $< S^G >$.

If $g_1, \ldots, g_n$ are elements of a group G, then

$$[g_1, g_2] = g_1^{-1}g_2^{-1}g_1g_2 \text{ , and}$$

$$[g_1, g_2, \ldots, g_n] = [[g_1, \ldots, g_{n-1}], g_n].$$

Also, if $X_1, \ldots, X_n$ are subgroups of G, then

$$[X_1, X_2] = < [x_1, x_2] \; ; \; x_i \in X_i \; , \; i = 1,2 >,$$

and $[X_1, \ldots, X_n] = [[X_1, \ldots, X_{n-1}], X_n]$.

If π is a set of primes then π' denotes the complementary set, and if p is a prime, p' denotes the set of primes different from p. Suppose that π is a set of primes. An element x of finite order in a group is said to be a π-element if the prime divisors of the order of x all lie in the set π. A group G is a π-group if every element of G is a π-element.

If $\mathfrak{X}$ is a class of groups (by which we mean that

$\mathfrak{X}$ contains all groups of order 1 and is closed under isomorphisms) and π is a set of primes, we denote by $\mathfrak{X}_\pi$ the class of π -groups in $\mathfrak{X}$, and by $\mathfrak{X}^*$ the class of finite $\mathfrak{X}$ -groups. $\mathfrak{S}$ will denote the class of periodic locally soluble groups (that is groups in which every element has finite order and every finitely generated subgroup is soluble), $\mathfrak{N}$ the class of nilpotent groups, and $\mathfrak{A}$ the class of abelian groups. Thus $\mathfrak{S}^*$, $\mathfrak{N}^*$ and $\mathfrak{A}^*$ are respectively the classes of finite soluble, finite nilpotent and finite abelian groups. We shall use, and assume, the notation of group classes and closure operations developed by P. Hall, and set out for example in 10 and 17.

A group is said to be <u>locally finite</u> if every finite set of elements generates a finite subgroup. $\mathfrak{S}$-groups are well known to be locally finite. In this thesis the word 'group' will always mean 'locally finite group', unless the contrary is explicitly stated.

If G is a group and π is a set of primes, by a <u>Sylow π -subgroup</u> of G we mean a maximal π -subgroup of G; Zorn's Lemma shows that every group has Sylow π -subgroups. If p is a prime, then a Sylow p'-subgroup of a group G will sometimes be refered to as a <u>Sylow p-complement</u> or just a <u>p -complement</u> of G.

The Hirsch - Plotkin theorem is well known, and so, by Zorn's Lemma, every group G possesses a unique maximal normal locally nilpotent subgroup, called the <u>radical</u> of G, and is denoted by $\rho(G)$.

The <u>upper locally nilpotent series</u>

$$1 = R_0 \leq R_1 \leq \ldots\ldots \leq R_\alpha \leq \ldots\ldots$$

of a group G is defined inductively by the rules

$R_0 = 1$, $R_{\alpha+1}/R_\alpha = \rho(G/R_\alpha)$, and if σ is a limit ordinal, then $R_\sigma = \bigcup_{\alpha<\sigma} R_\alpha$. Thus $G \in \acute{P}L\Pi$ if and only if there exists an ordinal γ such that $R_\gamma = G$.

Since the class of locally nilpotent groups is residually closed relative to the class of locally finite groups, every group G possesses a locally nilpotent residual (that is a unique normal subgroup minimal with respect to the factor group being locally nilpotent), denoted by $\sigma(G)$.

The <u>lower locally nilpotent series</u>

$$G = S_0 \geq S_1 \geq \ldots\ldots \geq S_\alpha \geq \ldots\ldots$$

of a group G is defined inductively by the rules

$S_0 = G$, $S_{\alpha+1} = \sigma(S_\alpha)$, and if σ is a limit ordinal, then $S_\sigma = \bigcap_{\alpha<\sigma} S_\alpha$. Thus $G \in \grave{P}L\Pi$ if and only if $S_\gamma = 1$ for some ordinal γ.

By a <u>Sylow basis</u> of a group G we mean a complete set $\underset{\sim}{S} = \{ S_p \}$ of Sylow p -subgroups S_p of G, one for each prime p, such that $< S_p \; ; \; p \in \pi >$ is a π-group for each set of primes π. If the subgroups S_p, in a Sylow basis $\underset{\sim}{S} = \{ S_p \}$, are pairwise permutable, we shall frequently write

$$S_\pi = < S_p \; ; \; p \in \pi >, \text{ and } S_{p'} = < S_q \; ; \; p \neq q >.$$

If T is a Sylow π-subgroup of a group G we say that T <u>reduces</u> into a subgroup H of G, if $T \cap H$ is a Sylow π-subgroup of H. If $\underset{\sim}{S} = \{ S_p \}$ is a Sylow basis of the group G, we say that $\underset{\sim}{S}$ <u>reduces</u> into H, if $\underset{\sim}{S} \cap H = \{ S_p \cap H \}$ is a Sylow basis of H. It is clear that $\underset{\sim}{S}$ reduces into H if and only if S_p reduces into H for all primes p.

An automorphism α of a group G is called a <u>normal automorphism</u> of G, if $N^{\alpha} = N$ for all normal subgroups N of G. We shall denote the collection of all normal automorphisms of G by Naut(G). An automorphism α of a group G is called a <u>locally inner automorphism</u> of G if, given any finite set of elements $g_1, \ldots, g_n$ of G, there exists an element $g \in G$, depending in general on the set in question, such that

$$g_i^{\alpha} = g^{-1} g_i g \ , \text{ for } i = 1, \ldots . n.$$

Two subgroups X and Y of G are said to be <u>locally conjugate</u> in G if there exists a locally inner automorphism of G mapping X onto Y. It is clear that a locally inner automorphism of a group G is a normal automorphism of G. However, the converse is not true, as is seen in the case of a cyclic group of order 3. The product of two locally inner automorphisms is again a locally inner automorphism. Also, if N is a normal subgroup of a group G, then a locally inner automorphism of G induces a locally inner automorphism in G/N in an obvious way. We shall denote the collection of all locally inner automorphisms of a group G by Linn(G).

A subgroup H of a group G is said to be <u>pronormal</u> in G if H and H^x are conjugate in $< H, H^x >$ for each element x of G. Let G be a group and let $\mathcal{C} = \{ N_\lambda \, ; \, \lambda \in \Lambda \}$ be a family of normal subgroups of G. A subgroup H of G is called a <u>distributive subgroup of G with respect to $\mathcal{C}$</u>, if $\bigcap_{\lambda \in \Lambda} HN_\lambda = H(\bigcap_{\lambda \in \Lambda} N_\lambda)$. H is called a <u>distributive subgroup</u> of G if H is a distributive subgroup of G with respect to $\mathcal{C}$, for all families $\mathcal{C}$ of normal subgroups of G.

Define a class of groups $\mathfrak{X}$ by the following:

$G \in \mathfrak{X}$ if and only if

(X1) the Sylow p-complements of N are pronormal in G, for all normal subgroups N of G,

(X2) $G \in \acute{P}L\Pi \cap \grave{P}L\Pi$,

(X3) G possesses Sylow bases,

(X4) if $\underset{\sim}{T}$ is a Sylow basis of a subgroup H of G, then there exists a Sylow basis $\underset{\sim}{S}$ of G such that $\underset{\sim}{S} \cap H = \underset{\sim}{T}$.

For each group G, let $A(G)$ be a subset of $\mathrm{Aut}(G)$, the group of automorphisms of G, such that

(A1) $A(G) \leqslant \mathrm{Naut}(G)$,

(A2) if H is a subgroup of G and $\beta \in A(H)$, then there exists $\alpha \in A(G)$ such that α agrees with β on H,

(A3) if N is a normal subgroup of G and $\beta \in A(G/N)$, then there exists $\alpha \in A(G)$ such that α induces β on G/N,

(A4) let H and K be distributive subgroups of G with respect to $\mathcal{C} = \{ S_\sigma ; \sigma < \gamma$ a limit ordinal$\}$, a subseries of the lower locally nilpotent series of G, and let $\alpha_\sigma \in A(G)$ such that $(HS_\sigma)^{\alpha_\sigma} = KS_\sigma$, for all $\sigma < \gamma$, then there exists $\alpha \in A(G)$ such that $(H(\bigcap_{\sigma<\gamma} S_\sigma))^{\alpha} = K(\bigcap_{\sigma<\gamma} S_\sigma)$ and given $h \in H$, there exists $\sigma < \gamma$ such that $h^{\alpha} = h^{\alpha_\sigma}$.

For example, let G be a periodic locally soluble FC-group, then Linn(G) satisfies (A1) - (A4); this can be found in 19. Clearly if G is a $\mathfrak{U}$-group, then Inn(G), the group of inner automorphisms of G, also satisfies (A1) - (A4).

Define $G \in (\mathfrak{X}, A)$ if and only if

(i) $G \in \mathfrak{X}$,

(ii) the Sylow structure(*) of G is permuted transitively by A(G).

(*) the Sylow structure of a group G is the collection of Sylow π-subgroups for all sets of primes π, and the collection of all Sylow bases of G.

Let $(\mathfrak{W}, A) = (\mathfrak{X}, A)^{QS}$ = the QS-interior of $(\mathfrak{X}, A)$, that is the largest QS-closed subclass of $(\mathfrak{X}, A)$.

When refering to $(\mathfrak{W}, A)$-groups we shall take it for granted that there is an underlying group of automorphisms satisfying (A1) - (A4), and just call them $\mathfrak{W}$-groups. An example of a $\mathfrak{W}$-group G which is not the direct product of a $\mathfrak{U}$-group and an FC-group is given in the Appendix A2; the group of automorphisms involved is Linn(G).

We shall show in section 1.1 that the set $\underset{\sim}{S} = \{ S_p \}$ of Sylow p-subgroups of a $\mathfrak{W}$-group G forms a Sylow basis of G if and only if they are pairwise permutable.

Let $\underset{\sim}{S} = \{ S_p \}$ be a Sylow basis of a $\mathfrak{W}$-group G, then the associated p-complement system of $\underset{\sim}{S}$ is the set $\{ S_{p'} \}$, where $S_{p'} = \langle S_q \,;\, q \neq p \rangle$. With this notation $D = \bigcap_p N_G(S_p) = \bigcap_p N_G(S_{p'})$ is the basis normalizer $N_G(\underset{\sim}{S})$ of $\underset{\sim}{S}$. The basis normalizers of a $\mathfrak{W}$-group G evidently form a unique A(G)-transitive class of subgroups of G, and we shall show in section 1.3 that they behave very much like the system normalizers of a finite soluble group.

Let Ω be a totally ordered set and let G be an arbitrary group. By a <u>series of type Ω of G</u> we mean a set $\{ U_\sigma, V_\sigma ; \sigma \in \Omega \}$ of pairs of subgroups of G indexed by Ω and satisfying

(i) V_σ is a normal subgroup of U_σ, for all $\sigma \in \Omega$,

(ii) $U_\alpha \leq V_\sigma$, if $\alpha < \sigma$,

(iii) $G - 1 = \bigcup_{\sigma \in \Omega} (U_\sigma - V_\sigma)$,

where G - 1 denotes the set of non - identity elements of G and $U_\sigma - V_\sigma$ denotes the set of elements of U_σ which do not belong to V_σ. Such a series is called a <u>normal series</u> if the subgroups U_σ and V_σ are all normal subgroups of G, and is a <u>chief series</u> if in addition U_σ/V_σ is a minimal normal subgroup of G/V_σ for each $\sigma \in \Omega$. Every normal series can be refined to a chief series, but in general Jordan - Hölder theorems do not hold for series of this kind. Every chief factor of an $\mathfrak{S}$- (and hence $\mathfrak{W}$-) group is an elementary abelian p -group (possibly infinite) for some prime p, 16 or 17 4.31. We refer to 17 for a fuller discussion of series.

If H/K is a chief factor of an arbitrary group G we denote by $A_G(H/K)$ the group of automorphisms induced by G on H/K. Thus $A_G(H/K) \cong G/C_G(H/K)$. If G is a periodic locally soluble group, p is a prime, and $\mathfrak{Y}$ is some class of groups, we denote by $C_G(\mathfrak{Y}, p)$ the intersection of the centralizers in G of those p -chief factors H/K of G for which $A_G(H/K) \in \mathfrak{Y}$. This group is called the

<u>$(\mathfrak{Y}, p)$ -centralizer of G</u>.

Let $\mathfrak{D}$ be a Q -closed subclass of $\mathfrak{S}$. A subclass $\mathfrak{X}$ of $\mathfrak{D}$ is called a <u>$(\mathfrak{D}, p)$ -preformation</u> if the following two conditions are satisfied:

P1. $\mathfrak{X} = Q\mathfrak{X}$,

P2. if $G \in \mathfrak{D}$ then $G/C_G(\mathfrak{X}, p) \in \mathfrak{X}$.

These conditions are automatically satisfied if $\mathfrak{X}$ is a <u>$\mathfrak{D}$ -formation</u>, that is a Q -closed subclass of $\mathfrak{D}$ such that $\mathfrak{D} \cap R\mathfrak{X} = \mathfrak{X}$.

Let $\mathfrak{K}$ be a QS -closed subclass of $\mathfrak{W}$; we obtain saturated $\mathfrak{K}$ -formations as follows. If π is a non - empty set of primes, a <u>$\mathfrak{K}$ -preformation function f on π</u> associates with each $p \in \pi$ a $(\mathfrak{K}, p)$ -preformation f(p). The <u>saturated $\mathfrak{K}$ -formation</u> defined by f is

$$\mathfrak{F} = \mathfrak{F}(f) = \mathfrak{K} \cap \mathfrak{S}_\pi \cap \bigcap_{p\in\pi} \mathfrak{S}_{p'}\mathfrak{S}_p f(p) .$$

We show in section 1.3 that $\mathfrak{F}$ is in fact a $\mathfrak{K}$ -formation and consists of all π -groups G in $\mathfrak{K}$ such that for all $p \in \pi$ and p -chief factors H/K of G, $A_G(H/K) \in f(p)$.

A subgroup H of a group G is said to <u>cover</u> the section U/V of G if $(H \cap U)V = U$, and to <u>avoid</u> U/V if $H \cap U = H \cap V$.

If $\mathfrak{X}$ is any class of groups, an <u>$\mathfrak{X}$ -projector</u> of a group G is an $\mathfrak{X}$ -subgroup X of G such that whenever $X \leqslant H \leqslant G$, K is a normal subgroup of H and $H/K \in \mathfrak{X}$, then $H = KX$. $L\mathfrak{N}$, the class of all locally nilpotent groups, is the saturated $\mathfrak{W}$-formation defined by the $\mathfrak{W}$-formation function f(p) = 1, on the set of all primes. The $L\mathfrak{N}$ -projectors will be called the <u>Carter subgroups</u>, but in $\mathfrak{W}$-groups generally (unlike finite soluble groups) they are not characterized as the self - normalizing locally nilpotent subgroups.

If G is an arbitrary group, a subgroup H of G is said to be <u>abnormal</u> in G if $x \in \langle H, H^x \rangle$ for each element x in G. H is said to be <u>quasi-abnormal</u> in G if every subgroup of G containing H is self - normalizing in G.

Let H be a subgroup of an arbitrary group G, and let Ω be a totally ordered set. By a <u>series of type Ω from H to G</u> we mean a set $\{ U_\sigma , V_\sigma \; ; \sigma \in \Omega \}$ of pairs of subgroups of G containing H and satisfying

(i) V_σ is a normal subgroup of U_σ, for all $\sigma \in \Omega$,

(ii) $U_\alpha \leqslant V_\sigma$, if $\alpha < \sigma$,

(iii) $G - H = \bigcup_{\sigma \in \Omega} (U_\sigma - V_\sigma)$.

We shall say that H is <u>serial</u> in G, and write H ser G, if there is some series from H to G. This concept is a rather far reaching generalization of subnormality.

A collection Σ of subgroups of a group G is called a <u>local system of subgroups</u> of G if the following two conditions are satisfied:

(i) if $g \in G$, then there exists $H \in \Sigma$ such that $g \in H$,

(ii) if H and K belong to Σ, then there exists $L \in \Sigma$ such that $H \leqslant L$ and $K \leqslant L$.

Let $\mathcal{S}$ be a partially ordered collection of finite non - empty sets A_α , A_β , A_γ, ... satisfying the following three conditions:

(i) for each pair of sets A_α, $A_\beta \in \mathcal{S}$, there exists a third set $A_\gamma \in \mathcal{S}$ such that $A_\alpha \leqslant A_\gamma$, and $A_\beta \leqslant A_\gamma$,

(ii) for each pair A_α, $A_\beta \in \mathcal{S}$, with $A_\alpha \leq A_\beta$, there exists a single valued mapping $p_{\beta\alpha}$, called a <u>projection</u>, of A_β into A_α, and $p_{\alpha\alpha}$ is the identity mapping of A_α onto itself,

(iii) if $A_\alpha \leq A_\beta \leq A_\gamma$, then $p_{\gamma\alpha}$ is the product of $p_{\gamma\beta}$ and $p_{\beta\alpha}$.

A set $\mathcal{P}$ consisting of elements selected from some of the sets A_α is called a <u>projection set</u> if any two elements of $\mathcal{P}$ have a common inverse image in $\mathcal{P}$. The set consisting of a single element of one of the A_α is a projection set. Similarly, the set consisting of an element of A_γ and its images in all sets A_α, such that $A_\alpha \leq A_\gamma$, is a projection set. It is clear that a projection set can contain at most one element from any one set A_α. For, the projections are single valued. In the case where $\mathcal{P}$ contains exactly one element from each set A_α of $\mathcal{S}$, $\mathcal{P}$ is called a <u>complete</u> projection set.

The following result is due to Kurosh: it can be found in 14 p.167.

<u>Theorem</u>. Every projection set $\mathcal{P}$ of the system $\mathcal{S}$ is part of some complete projection set.

The above theorem will be used repeatedly in part two of this thesis.

If G is a group, then $O_{p',p}(G)$ denotes the largest normal $\mathfrak{S}_{p'}\mathfrak{S}_p$-subgroup of G. If α is a normal automorphism of G and N is a normal subgroup of G, then α^* will often be used to denote the normal automorphism which α induces naturally in G/N.

PART ONE

FORMATION THEORY IN $\mathcal{W}$-GROUPS.

1.1. SYLOW THEORY.

We begin with some elementary but useful results on Sylow π-subgroups and Sylow bases of $\mathfrak{W}$-groups.

Lemma 1.1.1. If N is a normal subgroup of a $\mathfrak{W}$-group G and S is a Sylow π-subgroup of G, then

(i) $S \cap N$ is a Sylow π-subgroup of N, and all Sylow π-subgroups of N are of this form,

(ii) SN/N is a Sylow π-subgroup of G/N, and all Sylow π-subgroups of G/N are of this form,

(iii) if T is a Sylow π'-subgroup of G then G = ST.

Proof. (i) Suppose, for a contradiction, that $S \cap N$ is properly contained in a Sylow π-subgroup D of N, then there exists a Sylow π-subgroup U of G such that $D \leqslant U$.

Now, there exists $\alpha \in A(G)$ such that $S = U^{\alpha}$, and so

$$D^{\alpha} \leqslant S.$$

Therefore $D^{\alpha} \leqslant S \cap N$ and hence $D^{\alpha} = S \cap N$, which is a contradiction.

Hence $S \cap N$ is a Sylow π-subgroup of N. Conversely, let V be a Sylow π-subgroup of N, then there exists a Sylow π-subgroup S of G such that $V \leqslant S$. Then clearly $V = S \cap N$.

(ii) We first consider the case when G/N is a π-group, proving that then G = SN.

Let $1 = R_0 \leqslant R_1 \leqslant \ldots\ldots \leqslant R_\sigma \leqslant \ldots\ldots \leqslant R_\rho = G$ be the upper locally nilpotent series of G.

We shall show by induction on σ that $R_\sigma \leqslant SN$.

If $\sigma = 0$, then trivially $R_0 \leq SN$.
Suppose it has been shown that for all $\gamma < \sigma$, $R_\gamma \leq SN$.

<u>Case (a)</u>. $\sigma - 1$ exists. Assume that there exists $x \in R_\sigma$, such that $x \notin SN$. Since G/N is a π-group we may choose x to be a π-element. Let L be a subgroup of R_σ containing $R_\sigma \cap SN$ and maximal with respect to $x \notin L$, and let $M = \langle L, x \rangle$.

Then L is a maximal subgroup of M, and since $M/R_{\sigma-1}$ is a locally nilpotent group, it follows that L is a normal subgroup of M (16 or 17 Theorem 4.31).

Now, since the Sylow π-subgroups of M are permuted transitively by $A(M)$, there exists $\alpha \in A(M)$ such that $x^\alpha \in S \cap R_\sigma \leq L$, since by (i) $S \cap R_\sigma$ is a Sylow π-subgroup of R_σ.
Therefore $x \in L^{\alpha^{-1}} = L$, which is a contradiction.

Hence $R_\sigma \leq SN$.

<u>Case (b)</u>. σ is a limit ordinal. Then $R_\sigma = \bigcup_{\gamma<\sigma} R_\gamma$.
But, by induction $R_\gamma \leq SN$, for all $\gamma < \sigma$.

Therefore $R_\sigma \leq SN$.

Hence by transfinite induction $G = SN$.
In the general case if N is any normal subgroup of G and H/N is a Sylow π-subgroup of G/N, then $H = TN$ for any Sylow π-subgroup T of H.
Suppose that T is contained in a Sylow π-subgroup V of G, then $TN = VN$, and hence $T = V$. So there exists $\alpha \in A(G)$ such that $S = T^\alpha$. Then

$$(TN/N)^{\alpha^*} = T^\alpha N/N = SN/N.$$

Hence SN/N is a Sylow π-subgroup of G/N.

(iii) Let $1 = R_0 \leq R_1 \leq \ldots \leq R_\sigma \leq \ldots \leq R_\rho = G$

be the upper locally nilpotent series of G. We shall show by induction on σ that $R_\sigma \leq ST$.

If $\sigma = 1$, then clearly $R_1 \leq ST$.

Suppose it has been shown that for all $\gamma < \sigma$, $R_\gamma \leq ST$.

<u>Case (a)</u>. $\sigma - 1$ exists. Then $R_{\sigma-1} \leq ST$.

Now, by (ii), $(S \cap R_\sigma)R_{\sigma-1}/R_{\sigma-1}$ is a Sylow π-subgroup of $R_\sigma/R_{\sigma-1}$, and $(T \cap R_\sigma)R_{\sigma-1}/R_{\sigma-1}$ is a Sylow π'-subgroup of $R_\sigma/R_{\sigma-1}$.

But $R_\sigma/R_{\sigma-1}$ is a locally nilpotent group, and so

$$(S \cap R_\sigma)R_{\sigma-1}(T \cap R_\sigma) = R_\sigma.$$

Therefore $R_\sigma \leq ST$.

<u>Case (b)</u>. σ is a limit ordinal. Then $R_\sigma = \bigcup_{\gamma<\sigma} R_\gamma$.

But, by induction $R_\gamma \leq ST$, for all $\gamma < \sigma$, therefore $R_\sigma \leq ST$.

Hence by transfinite induction $G = ST$.

$\square$

<u>Lemma 1.1.2</u>. Let H and K be subgroups of a $\mathfrak{W}$-group G such that $K \leq N_G(H)$, and let S be a π-subgroup of G such that $S \cap H$ and $S \cap K$ are Sylow π-subgroups of H and K respectively. Then $S \cap HK = (S \cap H)(S \cap K)$ and is a Sylow π-subgroup of HK.

<u>Proof</u>. Clearly $S \cap K$ normalizes $S \cap H$, and so

$$T = (S \cap H)(S \cap K)$$

is a π-subgroup of HK.

Now, $T \cap H = S \cap H$ is a Sylow π-subgroup of H, and the restriction to K of the homomorphism $\varphi : HK \to HK/H$ is an epimorphism.

Therefore φ maps $S \cap K$ onto a Sylow π-subgroup of HK/H, by Lemma 1.1.1 (ii).

Therefore $H(S \cap K)/H = TH/H$ and is a Sylow π-subgroup of HK/H.

Hence T is a Sylow π-subgroup of HK, and in particular $T = S \cap HK$.

$\square$

Lemma 1.1.3. Let N be a normal subgroup of a $\mathfrak{W}$-group G and let $\underset{\sim}{S}$ be a Sylow basis of G. Then,

(i) $\underset{\sim}{S}$ reduces into N,

(ii) $\underset{\sim}{S}N/N$ is a Sylow basis of G/N,

(iii) if H and K are subgroups of G with $K \leq N_G(H)$, and if $\underset{\sim}{S}$ reduces into both H and K, then $\underset{\sim}{S}$ reduces into HK,

(iv) $\prod_{p\in\pi} S_p$ is a Sylow π-subgroup of G.

Proof. (i) and (ii) are immediate from the definition and Lemma 1.1.1.

(iii) is immediate from Lemma 1.1.2.

(iv) Let $1 = R_0 \leq R_1 \leq \dots \leq R_\sigma \leq \dots \leq R_\rho = G$ be the upper locally nilpotent series of G.

We shall show by induction on σ that $R_\sigma \leq \prod_p S_p$, and hence show that $G = \prod_p S_p$.

If $\sigma = 0$, then trivially $R_0 \leq \prod_p S_p$.

Suppose it has been shown that for all $\gamma < \sigma$, $R_\gamma \leq \prod_p S_p$.

Case (a). $\sigma - 1$ exists. Now $R_\sigma/R_{\sigma-1}$ is a locally nilpotent group, and by (i) and (ii) $(\underset{\sim}{S} \cap R_\sigma)R_{\sigma-1}/R_{\sigma-1}$ is a Sylow basis of it.

Therefore $\prod_p (S_p \cap R_\sigma)R_{\sigma-1} = R_\sigma \leq \prod_p S_p$.

Case (b). σ is a limit ordinal. Then $R_\sigma = \bigcup_{\gamma<\sigma} R_\gamma$.

But, by induction $R_\gamma \leq \prod_p S_p$, for all $\gamma < \sigma$, therefore $R_\sigma \leq \prod_p S_p$.

Hence by transfinite induction $G = \prod_p S_p$.

Let $S = \prod_{p \in \pi} S_p$ and $T = \prod_{q \in \pi'} S_q$, then $G = ST$.

Therefore S is a Sylow π-subgroup of G.

□

Corollary 1.1.4. If N is a normal subgroup of a $\mathfrak{W}$-group G, then

(i) every Sylow basis of N has the form $\underset{\sim}{S} \cap N$ for some Sylow basis $\underset{\sim}{S}$ of G,

(ii) every Sylow basis of G/N has the form $\underset{\sim}{S}N/N$ for some Sylow basis $\underset{\sim}{S}$ of G.

Proof. This follows immediately from the above lemma.

□

The following lemma allows us to obtain an equivalent form of the definition of Sylow bases in $\mathfrak{W}$-groups.

Lemma 1.1.5. Let G be a $\mathfrak{W}$-group, σ, τ disjoint sets of primes, and let S, T be σ-, τ-subgroups respectively of G. Then the following statements are equivalent:

(i) $\langle S, T \rangle$ is a $\sigma \cup \tau$-group, and S and T are Sylow σ- and τ-subgroups respectively of it,

(ii) $ST = TS$.

Proof. Assume first (i). Clearly we may also assume that $G = \langle S, T \rangle$. As G is a $\sigma \cup \tau$-group, every σ'-element of G is a τ-element. Hence T is a Sylow σ'-subgroup of G.

Hence $G = ST$, by Lemma 1.1.1.

Conversely, we have $G = ST$, S a σ-group and T a σ'-group.

Therefore T is a Sylow σ'-subgroup of G.

Every σ'-element of G is the image (under some element of $A(G)$) of an element of T, and so is a τ-element.

Also every element x of G has the form

$$x = ab = ba,$$

where a is a σ-element and b is a σ'-element.

Therefore x is a $\sigma\cup\tau$-element, and hence G is a $\sigma\cup\tau$-group.

The result follows.

$\square$

<u>Corollary 1.1.6.</u> Let $\underset{\sim}{S} = \{ S_p \}$ be a complete set of Sylow p-subgroups of a $\mathcal{W}$-group G, one for each prime p. Then the following statements are equivalent:

(i) $\underset{\sim}{S}$ is a Sylow basis of G,

(ii) $S_pS_q = S_qS_p$, for all primes p and q.

<u>Proof.</u> Assume first (i). Then we have that $< S_p, S_q >$ is a $\{ p,q \}$-group, and S_p, S_q are Sylow p-, q-subgroups respectively of it.

Therefore $S_pS_q = S_qS_p$, by Lemma 1.1.5.

In proving (ii) implies (i) we must show that

$$H = \prod_{p\in\pi} S_p = < S_p;\ p \in\pi> \text{ is a } \pi\text{-group.}$$

Since any element of H lies in the product of finitely many S_p's, we may assume that π is finite, and use induction on the order of π .

If π is empty, then the above is trivial.

If not, write $\pi = \sigma\cup\{p\}$, where $p \notin \sigma$.

Then $\prod_{q\in\pi} S_q = S_\sigma S_p$, where $S_\sigma = \prod_{q\in\sigma} S_q$.
By induction S_σ is a σ-group. Therefore by Lemma 1.1.5

$$\prod_{q\in\pi} S_q \text{ is a } \sigma\cup\{p\}\text{-group.}$$

□

Lemma 1.1.7. Let H and M be normal subgroups of a $\mathcal{W}$-group G, let S,T be Sylow p'-, p-subgroups of G respectively, and let $N = N_G(S \cap M)$. Then:

(i) $HM \cap HS = H(M \cap S)$,

(ii) $HN = N_G(HM \cap HS)$,

(iii) T reduces into N.

Proof. (i) By Lemma 1.1.1, $S \cap M$ is a Sylow p'-subgroup of M, and $(S \cap M)H/H$ is a Sylow p'-subgroup of HM/H. Since $(HM \cap HS)/H$ is a p'-group, we have

$$HM \cap HS = H(M \cap S).$$

(ii) By (i) $N_G(HM \cap HS) = N_G(H(M \cap S))$.
Let $x \in N_G(H(M \cap S))$, then $(M \cap S)^x \leq H(M \cap S)$. Now, both $M \cap S$ and $(M \cap S)^x$ are Sylow p'-subgroups of M, and M is a normal subgroup of G. Hence by the pronormality axiom, there exists $y \in H$ such that

$$(M \cap S)^y = (M \cap S)^x.$$

Therefore $xy^{-1} \in N$, and so $x \in HN$.
Therefore $N_G(H(M \cap S)) \leq HN$, the converse is clear.

(iii) As $S \leq N$ and $G = ST$, we have $N = S(T \cap N)$.
Hence $T \cap N$ is a Sylow p-subgroup of N.

□

We now introduce a concept similar to but rather more general than that of the $\mathfrak{F}$-normalizers of a group corresponding to a saturated formation $\mathfrak{F}$. This notion was considered in the $\mathcal{U}$-group case in $\underset{\sim}{6}$.

If π is a set of primes and G is a $\mathcal{W}$-group, then by a <u>π-system</u> of G we mean a collection $\underset{\sim}{M} = \{ M_p \}$ for p in π, of normal subgroups of G, one for each $p \in \pi$.

If $\underset{\sim}{S}$ is a Sylow basis of G, then the <u>$\underset{\sim}{M}$-normalizer of G associated with $\underset{\sim}{S}$</u> is

$$D = S_\pi \cap \bigcap_{p \in \pi} N_p,$$

where $N_p = N_G(S_{p'} \cap M_p)$.

By hypothesis a $\mathcal{W}$-group G has a unique A(G)-transitive class of $\underset{\sim}{M}$-normalizers for each $\underset{\sim}{M}$. The following lemma gives some of their properties.

<u>Lemma 1.1.8</u>. Let $\underset{\sim}{M}$ be a π-system of a $\mathcal{W}$-group G, let $\underset{\sim}{S}$ be a Sylow basis of G, and let D be the $\underset{\sim}{M}$-normalizer of G associated with $\underset{\sim}{S}$. Then

(i) for $p \in \pi$, $S_p \cap N_p = S_p \cap D$ is a Sylow p-subgroup of D,

(ii) $\underset{\sim}{S}$ reduces into D,

(iii) $D = \langle S_p \cap N_p;\ p \in \pi \rangle = \langle S_p \cap D;\ p \in \pi \rangle$,

(iv) if H is a normal subgroup of G then DH/H is the $\underset{\sim}{M}H/H$-normalizer of G/H associated with $\underset{\sim}{S}H/H$.

<u>Proof</u>. (i) If $p, q \in \pi$ and $p \neq q$ then $S_p \leq S_{q'} \leq N_q$. Hence $S_p \cap N_p \leq S_\pi \cap \bigcap_{q \in \pi} N_q = D$. By Lemma 1.1.7, $S_p \cap N_p$ is a Sylow p-subgroup of N_p.

Therefore $S_p \cap N_p$ is a Sylow p-subgroup of D and

$$S_p \cap N_p = S_p \cap D.$$

(ii) and (iii) are immediate consequences of (i).

(iv) Since $S_p \cap N_p$ is a Sylow p-subgroup of N_p, $(S_p \cap N_p)H/H$ is a Sylow p-subgroup of N_pH/H.

Hence $(S_p \cap N_p)H/H = S_pH/H \cap N_pH/H$

$= S_pH/H \cap N_{G/H}(S_{p'} \cap M_pH/H)$, by Lemma 1.1.7.

Therefore by (iii) the $\underset{\sim}{M}H/H$ -normalizer of G/H associated with $\underset{\sim}{S}H/H$ is

$$< (S_p \cap N_p)H/H \; ; \; p \in \pi >.$$

But this is clearly DH/H.

$\square$

1.2. CHIEF FACTORS.

We will require the following result 6 Corollary 3.3.

Lemma 1.2.1. If G is a periodic locally soluble group and H/K is a p-chief factor of G, then

$$O_p(G/C_G(H/K)) = 1.$$

The following important result was proved by Gardiner, Hartley and Tomkinson for $\mathfrak{U}$-groups 6 Lemma 3.1.

Lemma 1.2.2. Let M be a normal subgroup of a $\mathfrak{W}$-group G, S a Sylow p'-subgroup of G, $N = N_G(S \cap M)$ and let H/K be a chief factor of G. Then N covers H/K unless H/K is a p-chief factor not centralized by M, in which case N avoids H/K.

Proof. By Lemma 1.1.7, we may assume that $K = 1$.

Case (a). H is a p'-group. Then $H \leqslant S \leqslant N$.

Case (b). H is a p-group centralized by M. Then H normalizes every subgroup of M and in particular $S \cap M$.

$$\text{Therefore } H \leqslant N.$$

Case (c). H is a p-group not centralized by M.

$$\text{Let } C = C_G(H) \text{ and } C_1 = C \cap M.$$

Then C and C_1 are normal subgroups of G, $C_1 < M$, and $M/C_1 \cong MC/C$ is a normal subgroup of G/C.

By Lemma 1.2.1, $O_p(G/C) = 1$, hence $O_p(M/C_1) = 1$. Since M/C_1 is a $\mathfrak{W}$-group it has a non - trivial locally nilpotent radical, and therefore contains for some prime

q a non - trivial characteristic q -subgroup Q/C_1. By the above remarks $q \neq p$. Hence $Q \leq (S \cap M)C$.

Now clearly $[H \cap N, S \cap M] \leq H \cap S \cap M = 1$. Hence Q centralizes $H \cap N$. But $C_H(Q)$ is a normal subgroup of G, and as $Q \nleq C$, $C_H(Q) < H$, and so $C_H(Q) = 1$.

Therefore $H \cap N = 1$, as required.

□

The covering and avoidance properties of $\underset{\sim}{M}$ -normalizers may be deduced from Lemma 1.2.2 in the usual way.

<u>Theorem 1.2.3</u>. Let $\underset{\sim}{M}$ be a π-system of a $\mathcal{W}$-group G, $\underset{\sim}{S}$ a Sylow basis of G, and let D be the corresponding $\underset{\sim}{M}$ -normalizer of G. Then D avoids all chief factors of G except the p -chief factors centralized by M_p with $p \in \pi$ which it covers.

<u>Proof</u>. We have

$$D = S_\pi \cap \bigcap_{p \in \pi} N_p$$

where $N_p = N_G(S_{p'} \cap M_p)$.

Clearly D avoids all π'-chief factors of G. Now if $p \in \pi$, then a p -chief factor of G not centralized by M_p is already avoided by N_p, by Lemma 1.2.2, and is a fortiori avoided by D.

On the other hand a p -chief factor of G centralized by M_p is covered by N_p and hence also by $S_p \cap N_p$, which is a Sylow p -subgroup of N_p, by Lemma 1.1.7. But, by Lemma 1.1.8, $S_p \cap N_p \leq D$, and so D also covers the chief factor in question.

□

We will require the following result $\underset{\sim}{6}$ Lemma 3.6.

Lemma 1.2.4. Let $\mathcal{Y}$ be a QS -closed class of groups and let $\mathcal{X}$ be a $\mathcal{Y}$ -formation. Let G be a $\mathcal{Y}$ -group and suppose that X is an $\mathcal{X}$ -projector of G. Then

(i) X is a distributive subgroup of G,

(ii) if X* is another $\mathcal{X}$ -projector of G distinct from X such that X and X* are conjugate in $\langle X, X^* \rangle$ and $\{ U_\sigma , V_\sigma ; \sigma \in \Omega \}$ is a normal series of G, then there exists $\sigma \in \Omega$ such that

$$U_\sigma X = U_\sigma X^* \text{ and } V_\sigma X \neq V_\sigma X^* .$$

We are now in a position to show that the group N_p considered in Lemma 1.2.2 depends only on the set of p -chief factors of G centralized by M. Thus the $\underset{\sim}{M}$ -normalizers of G for a given π-system $\underset{\sim}{M}$ depend only on the centralizing properties of the subgroups in $\underset{\sim}{M}$.

Lemma 1.2.5. Let M and M* be normal subgroups of a $\mathcal{W}$-group G. Let S be a Sylow p'-subgroup of G, and let $N_p = N_G(S \cap M)$, $N_p^* = N_G(S \cap M^*)$. Then the following three conditions are equivalent:

(i) $N_p = N_p^*$,

(ii) the set of p -chief factors of G centralized by M and M* respectively are the same,

(iii) in some chief series of G the sets of p -chief factors centralized by M and M* respectively are the same.

Proof. It is immediate from Lemma 1.2.2 that (i) implies (ii), for the p-chief factors of G centralized by M (respectively M^*) are just those covered by N_p (respectively N_p^*).

Clearly (ii) implies (iii).

We finally prove that (iii) implies (i). Now M, M^* and MM^* centralize the same p-chief factors in some chief series Σ of G. By replacing M^* by MM^* if necessary we may suppose that $M \leq M^*$. Then $N_p^* \leq N_p$.

Let $\Sigma = \{ U_\sigma, V_\sigma ; \sigma \in \Omega \}$ and consider

$$\Sigma \cap M^* = \{ U_\sigma \cap M^*, V_\sigma \cap M^* ; \sigma \in \Omega \}.$$

If $1 \neq (U_\sigma \cap M^*)/(V_\sigma \cap M^*)$ then $(U_\sigma \cap M^*)/(V_\sigma \cap M^*)$ is G-isomorphic to $(U_\sigma \cap M^*)V_\sigma/V_\sigma$ which is a non-trivial normal subgroup of G/V_σ contained in U_σ/V_σ.

Therefore $(U_\sigma \cap M^*)/(V_\sigma \cap M^*)$ is G-isomorphic to U_σ/V_σ, and is a chief factor of G.

If $U_\sigma \cap M^*/V_\sigma \cap M^*$ is a p-chief factor, then it is centralized by M if and only if it is centralized by M^*.

Suppose, for a contradiction, that $N_p^* < N_p$, and let $x \in N_p$, $x \notin N_p^*$.

Then, $(S \cap M^*)^x \neq S \cap M^*$ are both Sylow p'-subgroups of M^*, and so are conjugate in the group they generate; furthermore they are $\mathfrak{S}_{p'}$-projectors of M^*.

Therefore, by Lemma 1.2.4, there exists a factor H/K of $\Sigma \cap M^*$ such that

$$H(S \cap M^*) = H(S \cap M^*)^x \qquad \text{..........(1)}$$

$$K(S \cap M^*) \neq K(S \cap M^*)^x \qquad \text{..........(2)}$$

From (1), there exists $u \in H$ such that

$$(S \cap M^*)^{xu} = S \cap M^* ,$$

and from (2) $u \notin K$.

Now, $xu \in N_p^* < N_p$, and so $u \in N_p$, since $x \in N_p$. Therefore

$$u \in H \cap N_p \text{ and } u \notin K.$$

Hence N_p does not avoid H/K, and so, by Lemma 1.2.2, H/K is either a p'-factor or a p -chief factor centralized by M. In the latter case M* also centralizes H/K by assumption.

Therefore in either case N_p^* covers H/K, and so $u \in KN_p^*$. Hence u normalizes K(S ∩ M*), and so

$$K(S \cap M^*)^x = K(S \cap M^*)^{u^{-1}} = K(S \cap M^*)$$

which is a contradiction.

□

We require the following result in the sequel 6 Theorem 3.8.

<u>Lemma 1.2.6</u>. Let G be a periodic locally soluble group. Then

$$O_{p'p}(G) = \bigcap C_G(H/K)$$

where the intersection may be taken either over all p -chief factors of G or over all those p -chief factors of G occurring in some chief series of G.

1.3. $\mathfrak{F}$- NORMALIZERS.

Henceforth $\mathcal{K}$ will denote a fixed QS-closed subclass of $\mathcal{W}$, π will denote a fixed set of primes, f will be a $\mathcal{K}$-preformation function on π, and $\mathfrak{F}$ will be the saturated $\mathcal{K}$-formation defined by f. Thus

$$\mathfrak{F} = \mathfrak{F}(f) = \mathcal{K} \cap \mathfrak{S}_{\pi} \cap \bigcap_{p\in\pi} \mathfrak{S}_{p'} \mathfrak{S}_{p} f(p).$$

Lemma 1.3.1. Let G be a $\mathcal{K}_{\pi}$-group, then the following statements are equivalent:

(i) G is an $\mathfrak{F}$-group,

(ii) $G/O_{p'p}(G) \in f(p)$, for all $p \in \pi$,

(iii) for all $p \in \pi$ and p-chief factors H/K of G, $A_G(H/K) \in f(p)$.

Proof. Clearly (i) and (ii) are equivalent.
Assume (ii). Then, by Lemma 1.2.6, we have

$$O_{p'p}(G) \leq C_G(H/K)$$

for any p-chief factor H/K of G. So

$$A_G(H/K) \cong G/C_G(H/K) \in Qf(p) = f(p).$$

Conversely, if every p-chief factor H/K of G satisfies $A_G(H/K) \in f(p)$ for all $p \in \pi$, then

$$C_G(\, f(p),\, p\,) = \bigcap C_G(H/K) = O_{p'p}(G),$$

by Lemma 1.2.6. Therefore

$$G/O_{p'p}(G) \in f(p), \text{ for all } p \in \pi.$$

□

If f is some preformation function and G some group, we shall often refer to the f(p) -centralizer of G rather than the (f(p), p) -centralizer and this group will often be denoted by C_p or $C_p(G)$.

The next result shows that in constructing the f(p) -centralizer of G it is sufficient to consider only those chief factors occurring in a given chief series of G.

<u>Lemma 1.3.2</u>. Let G be a $\mathcal{K}$-group, C_p be the f(p) -centralizer of G, and let Σ be a chief series of G with $C_p^* = \bigcap C_G(H/K)$ where the intersection is taken over all p -chief factors H/K in Σ such that $A_G(H/K) \in f(p)$. Then $C_p = C_p^*$.

<u>Proof</u>. Clearly $C_p \leq C_p^*$.
Now $G/C_p \in f(p)$. Let H/K be any p -chief factor of Σ centralized by C_p. Then $A_G(H/K) \in f(p)$, and so C_p^* also centralizes H/K.

Therefore C_p and C_p^* centralize the same p -chief factors in Σ, and hence, by Lemma 1.2.5, they centralize the same p -chief factors of G.

So, if U/V is any p -chief factor of G such that $A_G(U/V) \in f(p)$, then C_p centralizes U/V, hence C_p^* does so.

Therefore $C_p^* \leq C_G(U/V)$ and we have the reverse inequality as required.

$\square$

<u>Lemma 1.3.3</u>. $\mathfrak{F}$ is a $\mathcal{K}$ -formation.

Proof. That $\mathfrak{F} = Q\mathfrak{F}$ is clear, since
$Q(\mathfrak{S}_{p'}\mathfrak{S}_p f(p)) \leq (Q\mathfrak{S}_{p'})(Q\mathfrak{S}_p)(Qf(p)) = \mathfrak{S}_{p'}\mathfrak{S}_p f(p)$
for all $p \in \pi$.

So, let G be a $\mathcal{K}$-group and suppose that G has a family of normal subgroups $\{K_\lambda\}$, $\lambda \in \Lambda$ such that

$$G/K_\lambda \in \mathfrak{F} \quad \text{and} \quad \bigcap_{\lambda\in\Lambda} K_\lambda = 1.$$

We must show that G is an $\mathfrak{F}$-group. Clearly G is a $\mathcal{K}_\pi$-group. For $p \in \pi$ let C_p be the $f(p)$-centralizer of G. Then, by P2, $G/C_p \in f(p)$.

Now, as $G/K_\lambda \in \mathfrak{F}$, every p-chief factor H/K of G with $K_\lambda \leq K \leq H$ satisfies $A_G(H/K) \in f(p)$, by Lemma 1.3.1. Hence C_p centralizes every p-chief factor of G/K_λ.

Therefore, by Lemma 1.2.6,

$$C_pK_\lambda/K_\lambda \leq O_{p'p}(G/K_\lambda).$$

Therefore $C_pK_\lambda/K_\lambda \in \mathfrak{S}_{p'}\mathfrak{S}_p$.
Hence $C_p/C_p \cap K_\lambda \in \mathfrak{S}_{p'}\mathfrak{S}_p$. Since $\bigcap_{\lambda\in\Lambda} K_\lambda = 1$ and $\mathfrak{S}_{p'}\mathfrak{S}_p$ is an $\mathfrak{S}$-formation, it follows that

$$C_p \in \mathfrak{S}_{p'}\mathfrak{S}_p.$$

Therefore $G \in \mathfrak{S}_{p'}\mathfrak{S}_p f(p)$, and so G is an $\mathfrak{F}$-group.

$\square$

A $\mathcal{K}$-preformation function f on π is said to be <u>integrated</u> if $f(p) \leq \mathfrak{F}$, for all $p \in \pi$.

Corollary 1.3.4. Every saturated $\mathcal{K}$-formation can be defined by an integrated $\mathcal{K}$-preformation function.

Proof. It is easy to see that the intersection of two $(\mathcal{K}, p)$-preformations is another.

By Lemma 1.3.3, $\mathfrak{F}$ is a ($\mathcal{K}$, p) -preformation for all p, and so the function f* defined by

$$f^*(p) = \mathfrak{F} \cap f(p), \text{ for } p \in \pi,$$

is a $\mathcal{K}$ -preformation function on π.

Clearly $\mathfrak{F}(f^*) = \mathfrak{F}(f)$ and f* is certainly integrated.

□

We shall always assume that the $\mathcal{K}$ -preformation functions we consider are integrated.

<u>Lemma 1.3.5</u>. Let f and f* be two integrated $\mathcal{K}$ -preformation functions defining the same saturated $\mathcal{K}$ -formation $\mathfrak{F}$. Let G be a $\mathcal{K}$ -group, $p \in \pi$, and let H/K be a p -chief factor of G with centralizer C. Then $G/C \in f(p)$ if and only if $G/C \in f^*(p)$.

<u>Proof</u>. Suppose that $G/C \in f(p)$, let S be a Sylow p'-subgroup of C and let $N = N_G(S)$.

Let $x \in G$, then S^x is a Sylow p'-subgroup of C. Therefore there exists $y \in \langle S, S^x \rangle \leq C$ such that

$$S^{xy} = S$$

Therefore $xy \in N$, and so $x \in NC$. Hence $G = NC$.

Hence $G/C \cong N/(N \cap C)$.

Now, S is a Sylow p'-subgroup of $N \cap C$ and S is also a normal subgroup of $N \cap C$, so $(N \cap C)/S$ is a p -group.

We claim that $S = O_{p'}(N)$. For by Lemma 1.2.2, N covers H/K and so H/K is isomorphic to $(H \cap N)/(K \cap N)$ as N -groups.

$O_{p'}(N)$ centralizes $(H \cap N)/(K \cap N)$, and therefore it centralizes H/K.

Therefore $O_{p'}(N) \leq C$, and so $O_{p'}(N) = S$.

We also claim that $N/S \in \mathfrak{F}$. Clearly N/S is a π-group, $N/S \in \mathfrak{S}_p f(p)$, and if $p \neq q \in \pi$, then $N/S \in \mathfrak{S}_p f(p) \leq \mathfrak{S}_p \mathfrak{S}_{q'} \mathfrak{S}_q f(q) = \mathfrak{S}_{q'} \mathfrak{S}_q f(q)$.

Therefore $N/S \in \mathfrak{F}$.

But f^* also defines $\mathfrak{F}$, and so $N/S \in \mathfrak{S}_{p'} \mathfrak{S}_p f^*(p)$.

Since $O_{p'}(N) = S$, we have $O_{p'}(N/S) = 1$.

Therefore $N/S \in \mathfrak{S}_p f^*(p)$.

Hence $N/(N \cap C) \in \mathfrak{S}_p f^*(p)$, and so $G/C \in \mathfrak{S}_p f^*(p)$.

By Lemma 1.2.1, $O_p(G/C) = 1$, so $G/C \in f^*(p)$.

The result follows by symmetry.

$\square$

Suppose that $\mathfrak{F} = \mathfrak{F}(f)$ is a saturated $\mathfrak{K}$-formation, G is a $\mathfrak{K}$-group, and H/K is a p-chief factor of G. We say that H/K is <u>$\mathfrak{F}$-central</u> if $p \in \pi$ and $A_G(H/K) \in f(p)$; otherwise H/K is said to be <u>$\mathfrak{F}$-eccentric</u>.

By Lemma 1.3.5 above, these concepts depend only on $\mathfrak{F}$ and not on the particular preformation function defining $\mathfrak{F}$. With this terminology the $f(p)$-centralizer of G is the intersection of the centralizers of the $\mathfrak{F}$-central p-chief factors of G, and this too is independent of the way $\mathfrak{F}$ is defined.

Let C_p be the $f(p)$-centralizer of G. The $\underset{\sim}{C}$-normalizers of G corresponding to the π-system $\underset{\sim}{C} = \{ C_p \}$ for $p \in \pi$ are called the <u>$\mathfrak{F}$-normalizers</u> of G.

Since $\underset{\sim}{C}$ depends only on $\mathfrak{F}$ the $\mathfrak{F}$-normalizers of G also depend only on $\mathfrak{F}$. If $\underset{\sim}{S}$ is a Sylow basis of G, the $\mathfrak{F}$-normalizer of G associated with $\underset{\sim}{S}$ is thus

$$D = S_{\pi'} \cap \bigcap_{p\in\pi} N_p ,$$

where $N_p = N_G(S_{p'} \cap C_p)$.

It may be possible to recover the $\mathfrak{F}$-normalizers of G from π-systems of G other than $\underset{\sim}{C}$ as the following result shows.

<u>Lemma 1.3.6</u>. For $p \in \pi$, let C_p^* be a normal subgroup of a $\mathcal{K}$-group G such that $G/C_p^* \in f(p)$. Let $\underset{\sim}{S}$ be a Sylow basis of G, and let

$$D^* = S_\pi \cap \bigcap_{p\in\pi} N_G(S_{p'} \cap C_p^*), \; D = S_\pi \cap \bigcap_{p\in\pi} N_G(S_{p'} \cap C_p)$$

Then, $D^* \leq D$, and if $C_p^* \leq C_p$, for all $p \in \pi$, then $D = D^*$.

<u>Proof</u>. Let $C_p^{**} = C_p^* C_p$, and let H/K be any p-chief factor of G centralized by C_p^*.

Then $A_G(H/K) \in f(p)$, so H/K is $\mathfrak{F}$-central in G and is therefore centralized by C_p.

Therefore C_p^{**} centralizes H/K.

So, C_p^* and C_p^{**} centralize the same p-chief factors of G. Therefore, by Lemma 1.2.5, $N_G(S_{p'} \cap C_p^*) = N_G(S_{p'} \cap C_p^{**})$ for all $p \in \pi$.

Since $C_p^{**} \geq C_p$, it is clear that

$$N_G(S_{p'} \cap C_p^{**}) \leq N_G(S_{p'} \cap C_p).$$

Hence $D^* \leq D$.

If $C_p^* \leq C_p$, then the reverse inclusion is obvious, and so $D = D^*$ in this case.

□

Thus for example if the preformations f(p) are actually $\mathfrak{K}$-formations, then we can form the f(p)-residual R of a $\mathfrak{K}$-group G.

Since $G/C_p \in f(p)$, $R \leq C_p$. So the $\mathfrak{F}$-normalizers can be obtained as

$$D = S_\pi \cap \bigcap_{p \in \pi} N_G(S_{p'} \cap R).$$

Most of the basic properties of $\mathfrak{F}$-normalizers can be read off from results in sections 1.1 and 1.2. We summarize them in the following theorem.

<u>Theorem 1.3.7</u>. Let G be a $\mathfrak{K}$-group and let $\mathfrak{F}$ be a saturated $\mathfrak{K}$-formation. Then

(i) the $\mathfrak{F}$-normalizers of G are permuted transitively by A(G),

(ii) if K is a normal subgroup of G, then the $\mathfrak{F}$-normalizers of G/K are precisely the subgroups DK/K where D runs over the $\mathfrak{F}$-normalizers of G,

(iii) if $G \in \mathfrak{F}$, then the $\mathfrak{F}$-normalizers of G coincide with G,

(iv) if K is a normal subgroup of G, $G/K \in \mathfrak{F}$, and D is an $\mathfrak{F}$-normalizer of G, then G = DK,

(v) the $\mathfrak{F}$-normalizers of G cover the $\mathfrak{F}$-central chief factors of G and avoid the $\mathfrak{F}$-eccentric ones,

(vi) the $\mathfrak{F}$-normalizers of G belong to $\mathfrak{F}$.

<u>Proof</u>. (i) Since the Sylow bases of G are permuted transitively by A(G), the result follows from the definition of $\mathfrak{F}$-normalizers.

(ii) By Lemma 1.1.8, if D is the $\mathfrak{F}$-normalizer of

G associated with the Sylow basis $\underset{\sim}{S}$ of G, then DK/K is the normalizer of G/K associated with the Sylow basis $\underset{\sim}{S}K/K$ and the π-system $\underset{\sim}{C}K/K = \{ C_pK/K \}$ for $p \in \pi$.
Now,

$$(G/K)/(C_pK/K) \cong G/C_pK \in Qf(p) = f(p).$$

Also every $\mathfrak{F}$-central p-chief factor of G/K has the form $(U/K)/(V/K)$, where U/V is an $\mathfrak{F}$-central p-chief factor of G.

So, C_p centralizes U/V and so C_pK/K centralizes $(U/K)/(V/K)$. Thus

$$C_pK/K \leq C_p(G/K).$$

Therefore, by Lemma 1.3.6, DK/K is an $\mathfrak{F}$-normalizer of G/K. Since the $\mathfrak{F}$-normalizers of G/K are permuted transitively by $A(G/K)$ they all have this form.

(iii) If $G \in \mathfrak{F}$ then Lemma 1.2.6 shows that

$$C_p = O_{p'p}(G) \text{ for all } p \in \pi.$$

Therefore $S_{p'} \cap C_p$ is a normal subgroup of G, $N_p = G$, and

$$D = S_\pi \cap \bigcap_{p\in\pi} N_p = G.$$

(iv) This follows immediately from (ii) and (iii).

(v) Let D be an $\mathfrak{F}$-normalizer of G. Then by Theorem 1.2.3, D avoids all the chief factors of G except the p-chief factors centralized by C_p, for $p \in \pi$, which it covers.

The latter are the $\mathfrak{F}$-central ones.

(vi) Let D be the $\mathfrak{F}$-normalizer of G associated with the Sylow basis $\underset{\sim}{S}$ of G.

Now $G/C_p \in f(p) \leq \mathfrak{F}$, and so, by (iv), $G = C_pD$.
Therefore $D/(D \cap C_p) \cong DC_p/C_p = G/C_p \in f(p)$.

D normalizes $S_{p'} \cap C_p$, so it also normalizes $D \cap S_{p'} \cap C_p$. By Lemma 1.1.8, $\underset{\sim}{S}$ reduces into D, and so $S_{p'} \cap D$ is a Sylow p'-subgroup of D.

Therefore $D \cap S_{p'} \cap C_p$ is a Sylow p'-subgroup of $D \cap C_p$.

Hence $D \in \mathfrak{S}_{p'}\mathfrak{S}_p f(p)$, for all $p \in \pi$, and since $D \in \mathfrak{S}_\pi$ by definition, it follows that

$$D \in \mathfrak{F}.$$

$\square$

We now investigate the relationship between the $\mathfrak{F}$-normalizers of a $\mathcal{K}$-group G and the $\mathfrak{F}$-normalizers of certain subgroups of G. The following result on chief series of $\mathfrak{S}$-groups will be useful. It can be found in $\underset{\sim}{6}$ Lemma 4.7.

<u>Lemma 1.3.8</u>. Let G = RH be a periodic locally soluble group where $H \leq G$ and R is a normal locally nilpotent subgroup of G. Let $\{ U_\sigma, V_\sigma ; \sigma \in \Omega \}$ be any chief series of G. Then after suppression of trivial factors

$$\{ U_\sigma \cap H, V_\sigma \cap H ; \sigma \in \Omega \}$$

becomes a chief series of H, and if $(U_\sigma \cap H)/(V_\sigma \cap H)$ is non - trivial then

$$A_H((U_\sigma \cap H)/(V_\sigma \cap H)) \cong A_G(U_\sigma/V_\sigma).$$

<u>Theorem 1.3.9</u>. Suppose that G = RH is a $\mathcal{K}$-group where $H \leq G$ and R is a normal locally nilpotent subgroup of G. Let $\underset{\sim}{T}$ be a Sylow basis of H, $\underset{\sim}{R}$ the unique Sylow basis of R, and let $\underset{\sim}{S} = \{ R_p T_p \}$. Then $\underset{\sim}{S}$ is a Sylow

basis of G, and if D and D* are the $\mathfrak{F}$ -normalizers of G associated with $\underset{\sim}{S}$ and of H associated with $\underset{\sim}{T}$ respectively, then $D^* = D \cap H$.

<u>Proof</u>. Clearly R_pT_p is a Sylow p -subgroup of G, for all p, and

$$R_pT_pR_qT_q = R_qR_pT_pT_q = R_qT_qR_pT_p \text{ , for all p and q.}$$

Therefore $\underset{\sim}{S}$ is a Sylow basis of G.

We shall show that

$$C_p(G) \cap H \leqslant C_p(H) \text{ ....(1)}$$

where $C_p(H)$ and $C_p(G)$ are the f(p) -centralizers of H and G respectively, for $p \in \pi$.

Let Σ be any chief series of G. Then, by Lemma 1.3.8, if we intersect Σ with H we obtain a chief series Σ' of H every $\mathfrak{F}$ -central p -chief factor of which is H -isomorphic to some $\mathfrak{F}$ -central p -chief factor of Σ.

Hence $C_p(G) \cap H$ centralizes every $\mathfrak{F}$ -central p -chief factor in Σ'. By Lemma 1.3.2, $C_p(H)$ is the intersection of the centralizers in H of all these chief factors.

Therefore (1) is established.

Now,

$$D^* = T_\pi \cap \bigcap_{p\in\pi} N_H(\ T_{p'} \cap C_p(H)\ )$$
$$D\ = S_\pi \cap \bigcap_{p\in\pi} N_G(\ S_{p'} \cap C_p(G)\ ) \ .$$

From Lemma 1.2.6,

$$R \leqslant O_{p'p}(G) \leqslant C_p(G) \ .$$

Therefore

$$C_p(G) = R(\ H \cap C_p(G)\ ) \leqslant RC_p(H), \text{ by (1).}$$

Now, $S_{p'} = R_{p'}T_{p'}$; and $R_{p'}(\ T_{p'} \cap C_p(H)\ )$ is a Sylow

p' -subgroup of $RC_p(H)$.
Therefore $S_{p'} \cap RC_p(H) = R_{p'}(T_{p'} \cap C_p(H))$, which is normalized by D^*.

Therefore D^* normalizes $S_{p'} \cap C_p(G)$, and so $D^* \leq D$.

On the other hand $G = HC_p(G)$, since $R \leq C_p(G)$.

So

$$H/(H \cap C_p(G)) \cong G/C_p(G) \in f(p).$$

Therefore, by Lemma 1.3.6,

$$D^* = T_\pi \cap \bigcap_{p \in \pi} N_H(T_{p'} \cap H \cap C_p(G)) .$$

Now, by definition, D normalizes $S_{p'} \cap C_p(G)$, so $D \cap H$ normalizes

$$H \cap S_{p'} \cap C_p(G) = H \cap T_{p'} \cap C_p(G).$$

$S_\pi \cap H = T_\pi$, we therefore obtain $D \cap H \leq D^*$.

$\square$

We now investigate subgroups of G which contain an $\mathfrak{F}$ -normalizer of G.

<u>Theorem 1.3.10</u>. Let D be the $\mathfrak{F}$ -normalizer of the $\mathcal{K}$-group G associated with the Sylow basis $\underset{\sim}{S}$ of G, and suppose that $D \leq H \leq G$. Let $C_p(H)$, $C_p(G)$ be the $f(p)$ -centralizers of H and G respectively. Then

(i) $C_p(H) \leq C_p(G) \cap H$,

(ii) if $\underset{\sim}{S}$ reduces into H then D is contained in the $\mathfrak{F}$ -normalizer of H associated with $\underset{\sim}{S} \cap H$.

<u>Proof</u>. (i) Let L/M be any $\mathfrak{F}$ -central p -chief factor of G. We have to show that $C_p(H)$ centralizes L/M.

Now, by Theorem 1.3.7, D covers L/M, so H covers L/M.

Therefore

$$L/M \underset{H}{\cong} (L \cap H)/(M \cap H) \ldots\ldots\ldots\ldots(1)$$

Now $C_p(G)$ centralizes L/M, therefore $C_p(G) \cap H$ will centralize $(L \cap H)/(M \cap H)$.

But, since $G/C_p(G) \in f(p) \leqslant \mathfrak{F}$ and $D \leqslant H$, Theorem 1.3.7 gives $G = HC_p(G)$.

Therefore

$$H/(H \cap C_p(G)) \cong HC_p(G)/C_p(G) = G/C_p(G) \in f(p).$$

Therefore $A_H((L \cap H)/(M \cap H)) \in f(p)$, since $H \cap C_p(G)$ centralizes $(L \cap H)/(M \cap H)$.

Therefore $(L \cap H)/(M \cap H)$ is $\mathfrak{F}$-central in H, and so $C_p(H)$ centralizes it.

Therefore $C_p(H)$ centralizes L/M, by (1). Hence we have

$$C_p(H) \leqslant H \cap C_p(G).$$

(ii) For $p \in \pi$, D normalizes $S_{p'} \cap C_p(G)$, and since $D \leqslant H$, D also normalizes $S_{p'} \cap C_p(G) \cap H$.

It follows from (i) that D normalizes $S_{p'} \cap H \cap C_p(H)$. Hence D is contained in the $\mathfrak{F}$-normalizer of H associated with $\underset{\sim}{S} \cap H$.

$\square$

The following result extends Theorem 5.15 of Carter and Hawkes $\underset{\sim}{3}$.

<u>Theorem 1.3.11</u>. Let G be a $\mathfrak{K}$-group and suppose that the $\mathfrak{F}$-residual A of G is abelian. Then A is complemented in G and the complements are precisely the $\mathfrak{F}$-normalizers of G.

Proof. Now any complement of A in G belongs to $\mathfrak{F}$ and so is contained in some $\mathfrak{F}$-normalizer of G, by Theorem 1.3.9. It therefore remains only to show that every $\mathfrak{F}$-normalizer D of G complements A in G.

By Theorem 1.3.7, we have $G = AD$.

Let $\underset{\sim}{A} = \{ A_p \}$ be the unique Sylow basis of A, and let $L_{p'} = O_{p'}(D)$, for all $p \in \pi$.

Define $\varphi : G \to G/A_{p'}[A_p, L_{p'}] = G^*$, by

$$\varphi : x \to x^*.$$

Clearly $G^* \in \mathfrak{S}_{q'}\mathfrak{S}_q f(q)$, if $p \neq q \in \pi$.

On the other hand $L^*_{p'}$ is a normal subgroup of G^*, and $G^*/L^*_{p'} \in \mathfrak{S}_p f(p)$.

Therefore G^* is an $\mathfrak{F}$-group.

By the definition of A, $A_p = [A_p, L_{p'}]$. By Fitting's Lemma applied locally, we have

$$C_{A_p}(L_{p'}) = 1.$$

But $D \cap A_p$ is a normal p-subgroup of D, and so

$$[D \cap A_p, L_{p'}] = 1.$$

Hence $D \cap A_p = 1$, for all $p \in \pi$, and so we have

$$D \cap A = 1, \text{ as required.}$$

$\square$

1.4. $\mathfrak{F}$ -ABNORMAL SUBGROUPS.

Let G be a $\mathfrak{K}$ -group, then a maximal subgroup M of G is called <u>p -maximal</u> if M complements a p -chief factor of G.

Our first theorem shows that every maximal subgroup of a $\mathfrak{K}$ -group is p -maximal for some prime p. We require the following result [11] Lemma 2.3.

<u>Lemma 1.4.1</u>. Let X be an arbitrary locally nilpotent group and let A be a group of automorphisms of X. Suppose that whenever Y and B are finitely generated subgroups of X and A respectively the subgroup Y^B is finitely generated. Then every maximal A -invariant subgroup of X is a normal subgroup of X.

Notice that the hypotheses of the above lemma are satisfied if A is a locally finite group.

<u>Theorem 1.4.2</u>. Let M be a maximal subgroup of a periodic $\acute{P}L\Pi$-group, and let $K = \mathrm{core}_G(M)$. Then G/K has a unique minimal normal subgroup H/K. H/K is an elementary abelian p -group for some prime p and H/K complements M/K in G/K.

<u>Proof</u>. We may suppose without loss of generality that $K = 1$.

Let H be the locally nilpotent radical of G. Then $H \neq 1$ and so $H \nleq M$. Therefore $G = HM$.

Now $M \cap H$ is clearly a maximal M -invariant subgroup of H, and so, by Lemma 1.4.1, we have that $M \cap H$ is a normal subgroup of H.

Therefore $M \cap H$ is a normal subgroup of $MH = G$; and so

$$M \cap H \leq K = 1 .$$

It now follows that H is a minimal normal subgroup of G, and so is an elementary abelian p -group for some prime p, by a well - known theorem of D.H. Mclain 16 or 17 Theorem 4.31.

Since the locally nilpotent radical of G contains every minimal normal subgroup of G, H is the unique minimal normal subgroup of G.

$\square$

A p -maximal subgroup M of a $\mathfrak{K}$ -group G is called an <u>$\mathfrak{F}$ -normal maximal</u> subgroup of G, if $p \in \pi$ and $M/\mathrm{core}_G(M) \in f(p)$. Otherwise M is called an <u>$\mathfrak{F}$ -abnormal maximal</u> subgroup of G.

<u>Lemma 1.4.3</u>. Let G be a $\mathfrak{K}$ -group and suppose that M is a maximal subgroup of G. Then M is an $\mathfrak{F}$ -normal maximal subgroup of G if and only if M complements an $\mathfrak{F}$ -central chief factor of G.

<u>Proof</u>. Let $K = \mathrm{core}_G(M)$, and let H/K be the unique minimal normal subgroup of G/K; this exists by Theorem 1.4.2.

Let $C = C_G(H/K)$, then $H \leq C$.

Therefore $C = H(M \cap C)$, since $G = HM$.
Now $M \cap C$ is a normal subgroup of M, and

$$[M \cap C, H] \leq K \leq M \cap C.$$

Therefore $M \cap C$ is a normal subgroup of $MH = G$ and so $M \cap C \leq K$, i.e. $C = H$.
Therefore

$$A_G(H/K) \cong G/C = G/H = MH/H \cong M/(M \cap H) = M/\mathrm{core}_G(M).$$

Therefore $M/\mathrm{core}_G(M) \in f(p)$ if and only if $A_G(H/K) \in f(p)$ if and only if H/K is an $\mathfrak{F}$ -central chief factor of G.

$\square$

It is perhaps worth noting that in Theorem 1.4.2 H/K is self - centralizing in G/K. This is essentially the first part of the proof of Lemma 1.4.3.

It follows from Lemma 1.4.3. and Lemma 1.3.5. that the definitions of $\mathfrak{F}$ -normal maximal and $\mathfrak{F}$ -abnormal maximal depend only on $\mathfrak{F}$ and not on the particular integrated $\mathfrak{X}$ -preformation function f chosen to define $\mathfrak{F}$.

An arbitrary subgroup H of a $\mathfrak{X}$ -group G is called <u>$\mathfrak{F}$ -abnormal</u> in G if whenever $H \leq M < L \leq G$ and M is a maximal subgroup of L, then M is an $\mathfrak{F}$ -abnormal maximal subgroup of L.

In particular notice that G is an $\mathfrak{F}$ -abnormal subgroup of itself.

<u>Lemma 1.4.4</u>. Let G be a $\mathfrak{X}$ -group, then

(i) if H is $\mathfrak{F}$ -abnormal in G and $H \leq L \leq G$, then H is $\mathfrak{F}$ -abnormal in L and L is $\mathfrak{F}$ -abnormal in G,

(ii) if H is $\mathfrak{F}$-abnormal in G and N is a normal subgroup of G, then HN/N is $\mathfrak{F}$-abnormal in G/N.

Proof. Both follow immediately from the definition of $\mathfrak{F}$-abnormality.

$\square$

Let H be a subgroup of a $\mathfrak{K}$-group G and let $\mathfrak{F}$ be a saturated $\mathfrak{K}$-formation. Then H is called an $\mathfrak{F}$-residual supplement of G if $G = HF$, where F is the $\mathfrak{F}$-residual of G. H is called a radical supplement of G if $G = HR$, where R is the locally nilpotent radical of G.

Lemma 1.4.5. If E is an $\mathfrak{F}$-projector of a $\mathfrak{K}$-group G, then E is $\mathfrak{F}$-abnormal in G.

Proof. Suppose, for a contradiction, that the lemma is false.

Then there exist subgroups $E \leq M < L \leq G$ with M an $\mathfrak{F}$-normal maximal subgroup of L.
Let $K = \mathrm{core}_L(M)$, then, by Theorem 1.4.2, L/K has a unique minimal normal subgroup H/K and $L = MH$.

Since M is an $\mathfrak{F}$-normal maximal subgroup of L, H/K is a p-group for some $p \in \pi$ and H/K is $\mathfrak{F}$-central in L.

Therefore

$$A_L(H/K) \cong L/H \in f(p) \leq \mathfrak{F}.$$

Therefore L/K has a chief series passing through H/K in which every factor is $\mathfrak{F}$-central.

So, by Lemma 1.3.1, L/K is an $\mathfrak{F}$-group.

Hence, since E is an $\mathfrak{F}$-projector of L, we have

$$L = KE.$$

Therefore $L = M$, which is a contradiction.

$\square$

An $\mathfrak{F}$-abnormal radical supplement of G will be called an <u>$\mathfrak{F}$-critical</u> subgroup of G.

<u>Lemma 1.4.6</u>. Let H be an $\mathfrak{F}$-critical subgroup of a $\mathfrak{K}$-group G, let $\underset{\sim}{T}$ be a Sylow basis of H, $\underset{\sim}{S}$ a Sylow basis of G such that $\underset{\sim}{S} \cap H = \underset{\sim}{T}$, and let D and D^* be the $\mathfrak{F}$-normalizers of G associated with $\underset{\sim}{S}$ and H associated with $\underset{\sim}{T}$ respectively. Then $D = D^*$.

<u>Proof</u>. By Theorem 1.3.9, we have that $D^* = D \cap H$. Let R be the locally nilpotent radical of G. We shall show first that $D^* \cap R = D \cap R$.

Suppose, for a contradiction, that there exists an element x of $D \cap R$ which does not belong to $D^* \cap R$.

Let M be a subgroup of G containing H and maximal with respect to $x \notin M$, and let $L = \langle M, x \rangle$. Then M is a maximal subgroup of L.

Since $G = HR$, we have $G = MR = LR$. Therefore

$$L = (L \cap R)M.$$

It follows that $M \cap R$ is maximal among the M-invariant subgroups of $L \cap R$. So, by Lemma 1.4.1, $M \cap R$ is a normal subgroup of $L \cap R$.

Therefore $(L \cap R)/(M \cap R)$ is a chief factor of L. Now, since $\underset{\sim}{S}$ reduces into H and into every subgroup of R

it reduces into $L = H(L \cap R)$.

Therefore, by Theorem 1.3.9, $D^{**} = D \cap L$ is the $\mathfrak{F}$-normalizer of L associated with $\underset{\sim}{S} \cap L$.

Since $x \in D^{**} \cap L \cap R$ and $x \notin M \cap R$, D^{**} does not avoid $(L \cap R)/(M \cap R)$.

Therefore, by Theorem 1.3.7, $(L \cap R)/(M \cap R)$ is $\mathfrak{F}$-central in L.

Hence, by Lemma 1.4.3, M is $\mathfrak{F}$-normal in L, which contradicts the assumed $\mathfrak{F}$-abnormality of H.

Therefore

$$D^* \cap R = D \cap R.$$

Now, by Theorem 1.3.7, DR/R is the $\mathfrak{F}$-normalizer of G/R associated with $\underset{\sim}{S}R/R$, and D^*R/R is the $\mathfrak{F}$-normalizer of $HR/R = G/R$ associated with $\underset{\sim}{T}R/R = \underset{\sim}{S}R/R$.

Therefore

$$DR = D^*R .$$

Hence, since $D^* \leqslant D$, we have

$$D = D^*(D \cap R) = D^*(D^* \cap R) = D^* .$$

$\square$

<u>Lemma 1.4.7</u>. Let H be an $\mathfrak{F}$-abnormal $\mathfrak{F}$-residual supplement of a $\mathcal{K}$-group G, let $\underset{\sim}{S}$ be a Sylow basis of G which reduces into H, and let D be the $\mathfrak{F}$-normalizer of G associated with $\underset{\sim}{S}$. Then $D \cap H$ is contained in the $\mathfrak{F}$-normalizer of H associated with $\underset{\sim}{S} \cap H$.

<u>Proof</u>. Let K be a normal subgroup of G such that G/K is an $\mathfrak{F}$-group. Then $G = HK$.

In particular, for $p \in \pi$ we have $G = HC_p(G)$.

Therefore

$$H/(H \cap C_p(G)) \cong G/C_p(G) \in f(p).$$

Now,

$$D = S_\pi \cap \bigcap_{p \in \pi} N_G(S_{p'} \cap C_p(G)),$$

and so $D \cap H$ is contained in $S_\pi \cap H$ and normalizes $S_{p'} \cap H \cap C_p(G)$, for $p \in \pi$.

Therefore, by Lemma 1.3.6, it follows that $D \cap H$ is contained in the $\mathfrak{F}$-normalizer of H associated with $\underset{\sim}{S} \cap H$.

$\square$

<u>Question</u>. If G is an $\mathfrak{F}$-group, can G possess proper $\mathfrak{F}$-abnormal subgroups?

If the answer to the above question was in the negative, then we would be able to dispense with the restriction of being an $\mathfrak{F}$-residual supplement in the hypothesis of Lemma 1.4.7, and also in subsequent results where it is used. Unfortunately we have not been able to determine the answer to the above except under certain restrictions on $\mathfrak{F}$, as the following result indicates.

<u>Lemma 1.4.8</u>. Let $\mathfrak{F} \leq PL\mathfrak{N}$, and let G be an $\mathfrak{F}$-group. Then,

(i) if M and L are radical supplements of G with M maximal in L, then M is $\mathfrak{F}$-normal in L,

(ii) no proper $\mathfrak{F}$-abnormal subgroup of G is a radical supplement of G,

(iii) G has no proper $\mathfrak{F}$-abnormal subgroups.

<u>Proof</u>. (i) Let R be the locally nilpotent radical of G, then $G = LR$.

Let Σ be a chief series of G. Then every chief factor in Σ is $\mathfrak{F}$-central and it follows from Lemma 1.3.8. that $\Sigma \cap L$ is a chief series of L every factor of which is $\mathfrak{F}$-central in L. Therefore L is an $\mathfrak{F}$-group.

Since every chief factor of L is therefore $\mathfrak{F}$-central it follows that M complements an $\mathfrak{F}$-central chief factor of L.

Hence M is $\mathfrak{F}$-normal in L.

(ii) If X is a proper subgroup of G then there exist subgroups M and L of G with $X \leqslant M < L \leqslant G$ and M maximal in L. For G contains an element $x \notin X$; choose M containing X and maximal subject to $x \notin M$, and let $L = \langle M, x \rangle$.

If X is a radical supplement of G then so are L and M; therefore M is $\mathfrak{F}$-normal in L, by (i).

Hence X is not $\mathfrak{F}$-abnormal in G.

(iii) Suppose, for a contradiction, that X is a proper $\mathfrak{F}$-abnormal subgroup of G. Then let

$$1 = R_0 \leqslant R_1 \leqslant \ldots\ldots \leqslant R_n = G$$

be the upper locally nilpotent series of G, and let $i \geq 0$ be the smallest integer such that $XR_{i+1} = G$.

Then XR_i/R_i is a proper $\mathfrak{F}$-abnormal subgroup of G/R_i supplementing the locally nilpotent radical R_{i+1}/R_i of G/R_i.

Since G/R_i is an $\mathfrak{F}$-group, this is impossible by (ii)

□

Let H be a subgroup of a $\mathfrak{K}$-group G, then H is called a <u>complete $\mathfrak{F}$-residual supplement</u> of G if whenever $H \leq X \leq G$, then H supplements the $\mathfrak{F}$-residual of X.

<u>Lemma 1.4.9</u>. Let E be a subgroup of a $\mathfrak{K}$-group G, and let N be a normal subgroup of G contained in H such that E/N is an $\mathfrak{F}$-projector of G/N. Then E is a complete $\mathfrak{F}$-residual supplement of G.

<u>Proof</u>. Let $E \leq X \leq G$, and let F be the $\mathfrak{F}$-residual of X.

Then EF/NF is an $\mathfrak{F}$-projector of X/NF, and hence

$$EF = X.$$

$\square$

<u>Theorem 1.4.10</u>. Let H be an $\mathfrak{F}$-abnormal complete $\mathfrak{F}$-residual supplement of a $\mathfrak{K}$-group G, let $\underset{\sim}{S}$ be a Sylow basis of G which reduces into H, and let D be the $\mathfrak{F}$-normalizer of G associated with $\underset{\sim}{S}$. Then D is contained in the $\mathfrak{F}$-normalizer of H associated with $\underset{\sim}{S} \cap H$.

<u>Proof</u>. Let

$$1 = R_0 \leq \ldots. \leq R_\sigma \leq \ldots. R_\rho = G$$

be the upper locally nilpotent series of G. Then,

$$H = HR_0 \leq \ldots. \leq HR_\sigma \leq \ldots. HR_\rho = G$$

is an $\mathfrak{F}$-abnormal chain, and furthermore HR_σ/R_σ is $\mathfrak{F}$-critical in $HR_{\sigma+1}/R_\sigma$, for $0 \leq \sigma < \rho$.

Therefore there exists a series

$$H \leq H_1 \leq \ldots\ldots \leq H_\sigma \leq \ldots\ldots \leq H_\rho = G$$

such that there exists K_σ a normal subgroup of $H_{\sigma+1}$ with H_σ/K_σ an $\mathfrak{F}$-critical subgroup of $H_{\sigma+1}/K_\sigma$, for all $\sigma < \rho$ (just take $H_\sigma = HR_\sigma$, for all σ).

Now, $\underset{\sim}{S}$ reduces into H and also into R_σ, for all σ, and so, by Lemma 1.1.3, $\underset{\sim}{S}$ reduces into H_σ for all σ.

We shall prove the result by induction on σ.

If $\sigma = 0$, then the result trivially follows.

If $\sigma = 1$, then H/K is $\mathfrak{F}$-critical in G/K for some normal subgroup K of G contained in H.

Since $\underset{\sim}{S}K/K$ reduces into H/K, Lemma 1.4.6. shows that H/K contains the $\mathfrak{F}$-normalizer of G/K associated with $\underset{\sim}{S}K/K$. This $\mathfrak{F}$-normalizer is DK/K, and so $D \leq H$.

Therefore, by Lemma 1.4.7, D is contained in the $\mathfrak{F}$-normalizer of H associated with $\underset{\sim}{S} \cap H$.

Suppose that the result has been proved for all $\gamma < \sigma = \rho$.

<u>Case (a)</u>. $\sigma - 1$ exists. By the case $\sigma = 1$, we have that D is contained in the $\mathfrak{F}$-normalizer of $H_{\sigma-1}$ associated with $\underset{\sim}{S} \cap H_{\sigma-1}$. By the inductive hypothesis the latter $\mathfrak{F}$-normalizer is contained in that of H associated with $\underset{\sim}{S} \cap H$.

<u>Case (b)</u>. σ is a limit ordinal. Then $D = \bigcup_{\gamma<\sigma} (D \cap H_\gamma)$ and by Lemma 1.4.7, we have that $D \cap H_\gamma$ is contained in the $\mathfrak{F}$-normalizer D_γ of H_γ associated with $\underset{\sim}{S} \cap H_\gamma$, for all $\gamma < \sigma$.

But, by induction, we have that $D_\gamma \leq D_0$, for all $\gamma < \sigma$, where D_0 is the $\mathfrak{F}$-normalizer of H associated with $\underset{\sim}{S} \cap H$.

Therefore $D \leq \bigcup_{\gamma<\sigma} D_\gamma \leq D_0$.

$\square$

1.5. $\mathfrak{F}$ -PROJECTORS.

Before proving our main theorem we require a few basic results about distributive subgroups of $\mathcal{K}$ -groups. The first result gives several examples of such subgroups.

Lemma 1.5.1. Let G be a $\mathcal{K}$ -group, then the following are distributive subgroups of G:

(i) all Sylow π -subgroups of G, for all sets of primes π ,

(ii) if N is a normal subgroup of G and S is a Sylow π -subgroup of G, then $N_G(S \cap N)$ is distributive, for all sets of primes π ,

(iii) all $\mathfrak{F}$ -normalizers of G,

(iv) all $\mathfrak{F}$ -projectors of G.

Proof. (i) Let S be a Sylow π -subgroup of G, and let $\{ H_\lambda \}$ for $\lambda \in \Lambda$, be any family of normal subgroups of G.

Let $H = \bigcap_{\lambda\in\Lambda} H_\lambda$, then, by Lemma 1.1.1, SH/H is a Sylow π -subgroup of G/H.

Let $S^* = \bigcap_{\lambda\in\Lambda} (SH_\lambda)$, then $SH_\lambda = S^*H_\lambda$, for all $\lambda \in \Lambda$. Therefore

$$S^*/(S^* \cap H_\lambda) \cong S^*H_\lambda/H_\lambda = SH_\lambda/H_\lambda \cong S/(S \cap H_\lambda)$$

which is a π -group.

Therefore S^*/H is residually a π -group, and hence is a π -group.

But $S^* \geq SH$, and so $S^* = SH$ as required.

(ii) $\bigcap_{\lambda\in\Lambda} (H_\lambda N_G(N \cap S)) = \bigcap_{\lambda\in\Lambda} N_G(H_\lambda N \cap H_\lambda S)$, by Lemma 1.1.7. Let this subgroup be denoted by X. Then

$$X \leq N_G(\bigcap_{\lambda\in\Lambda}(H_\lambda N \cap H_\lambda S)) = N_G((\bigcap_{\lambda\in\Lambda} H_\lambda N) \cap (\bigcap_{\lambda\in\Lambda} H_\lambda)S), \text{ by (i)}$$
$$\leq N_G((\bigcap_{\lambda\in\Lambda} H_\lambda)N \cap (\bigcap_{\lambda\in\Lambda} H_\lambda)S)$$
$$= (\bigcap_{\lambda\in\Lambda} H_\lambda).N_G(N \cap S).$$

The reverse inequality is clear, and so we have the required result.

(iii) Let D be an $\mathfrak{F}$-normalizer of G. Since DH/H is an $\mathfrak{F}$-normalizer of G/H, by Theorem 1.3.7, we may assume that H = 1.

Let D be associated with the Sylow basis $\underset{\sim}{S}$ of G, then

$$D = S_\pi \cap \bigcap_{p\in\pi} N_G(S_{p'} \cap C_p(G)).$$

Therefore

$$\bigcap_{\lambda\in\Lambda}(DH_\lambda) = \bigcap_{\lambda\in\Lambda}\{H_\lambda(S_\pi \cap \bigcap_{p\in\pi} N_G(S_{p'} \cap C_p(G)))\}$$
$$\leq \bigcap_{\lambda\in\Lambda}\{H_\lambda S_\pi \cap \bigcap_{p\in\pi}(H_\lambda N_G(S_{p'} \cap C_p(G)))\}$$
$$= S_\pi \cap \bigcap_{p\in\pi}\bigcap_{\lambda\in\Lambda}(H_\lambda N_G(S_{p'} \cap C_p(G))), \text{ by (i)}$$
$$= S_\pi \cap \bigcap_{p\in\pi} N_G(S_{p'} \cap C_p(G)), \text{ by (ii)}$$
$$= D.$$

The reverse inequality is clear, hence

$$D = \bigcap_{\lambda\in\Lambda}(DH_\lambda).$$

(iv) Let E be an $\mathfrak{F}$-projector of G, and let $E^* = \bigcap_{\lambda\in\Lambda}(EH_\lambda)$. Then $E^*H_\lambda = EH_\lambda$, for all $\lambda \in \Lambda$.

Since EH/H is an $\mathfrak{F}$-projector of G/H, we may assume that H = 1. Then,

$$E^*/(E^* \cap H_\lambda) \cong E^*H_\lambda/H_\lambda = EH_\lambda/H_\lambda \cong E/(E \cap H_\lambda) \in Q\mathfrak{F} = \mathfrak{F}.$$

Therefore

$$E^* \in R\mathfrak{F} = \mathfrak{F}.$$

But, $E \leq E^*$, and so $E = E^*$, as required.

$\square$

Lemma 1.5.2. Let E be a subgroup of a $\mathcal{K}$ -group G, let S be a Sylow p -subgroup of G, and let $\{ R_\gamma ; \gamma < \sigma \}$ be members of the lower locally nilpotent series of G. If S reduces into ER_γ, for all $\gamma < \sigma$, and $E = \bigcap_{\gamma < \sigma} ER_\gamma$, then S reduces into E.

Proof. Suppose, for a contradiction, that $S \cap E$ is properly contained in a Sylow p -subgroup T of E.

Then, for all $\gamma < \sigma$, T is contained in a Sylow p -subgroup T_γ of ER_γ.

Therefore there exists $\beta_\gamma \in A(ER_\gamma)$ such that $T_\gamma^{\beta_\gamma} = S \cap ER_\gamma$, for all $\gamma < \sigma$. Hence there exists $\alpha_\gamma \in A(G)$ such that

$$(T_\gamma)^{\alpha_\gamma} = S \cap ER_\gamma, \text{ for all } \gamma < \sigma.$$

Therefore, by definition, there exists $\alpha \in A(G)$ such that $E^\alpha = E$. Hence α is an automorphism of E.

Let $t \in T$, then there exists $\gamma < \sigma$ such that

$$t^\alpha = t^{\alpha_\gamma}.$$

But, $t^{\alpha_\gamma} \in S \cap ER_\gamma \leqslant S$, and so $t^\alpha \in S \cap E$.

Hence $T^\alpha \leqslant S \cap E$, and so $S \cap E$ is a Sylow p -subgroup of E, which is a contradiction.

$\square$

Corollary 1.5.3. With the notation as in Lemma 1.5.2, if $\underset{\sim}{S}$ is a Sylow basis of G which reduces into ER_γ, for all $\gamma < \sigma$, then $\underset{\sim}{S}$ reduces into E.

In proving our main theorem we shall frequently require the particular $(L\mathfrak{N})\mathfrak{F}$ -case, in which the

$\mathfrak{F}$ -normalizers turn out to be precisely the $\mathfrak{F}$ -projectors sought.

Theorem 1.5.4. Let G be an $(L\mathfrak{N})\mathfrak{F} \cap \mathfrak{K}$ -group, then the $\mathfrak{F}$ -projectors of G are precisely the $\mathfrak{F}$ -normalizers of G.

Proof. Let R be a normal locally nilpotent subgroup of G such that G/R is an $\mathfrak{F}$ -group.

Let E be an $\mathfrak{F}$ -projector of G, then G = RE. Now, E is an $\mathfrak{F}$ -normalizer of itself, and so, by Theorem 1.3.9,

$$E \leqslant D$$

for some $\mathfrak{F}$ -normalizer D of G. But E covers D, and so

$$E = D.$$

Conversely, let D be any $\mathfrak{F}$ -normalizer of G, $D \leqslant H \leqslant G$, K a normal subgroup of H such that H/K is an $\mathfrak{F}$ -group, and suppose that D is associated with the Sylow basis $\underset{\sim}{S}$ of G.

By Theorem 1.3.7, G = RD, and by Lemma 1.1.8, $\underset{\sim}{S}$ reduces into D. Hence $S_p = R_p(S_p \cap D)$, where $\underset{\sim}{R} = \{ R_p \}$ is the unique Sylow basis of R.

Now, $H = (R \cap H)D$, and $\underset{\sim}{T} = \{ (R_p \cap H)(S_p \cap D) \}$ is a Sylow basis of H. Let D* be the $\mathfrak{F}$ -normalizer of H associated with $\underset{\sim}{T}$.

Then, by Theorem 1.3.9, $D^* = D \cap H$, i.e. $D^* = D$. Therefore D is an $\mathfrak{F}$ -normalizer of H, and so H = KD, by Theorem 1.3.7.

Thus D is an $\mathfrak{F}$ -projector of G.

$\square$

The following result will be useful [7] Hilfssatz 2.3.

Lemma 1.5.5. Let G be an arbitrary group, $\mathfrak{X}$ any Q-closed class, and let N be a normal subgroup of G. If X^*/N is an $\mathfrak{X}$-projector of G/N and X is an $\mathfrak{X}$-projector of X^*, then X is an $\mathfrak{X}$-projector of G.

Theorem 1.5.6. If G is a $\mathfrak{K}$-group, then G possesses $\mathfrak{F}$-projectors.

Proof. Let

$$G = R_0 \geq R_1 \geq \dots \geq R_\sigma \geq \dots \geq R_\rho = 1$$

be the lower locally nilpotent series of G.

We shall prove the existence of $\mathfrak{F}$-projectors in G/R_σ by induction on σ.

If $\sigma = 0$, then the above is trivial.

If $\sigma = 1$, then the result follows from Theorem 1.5.4.

Suppose that for all $\gamma < \sigma \leq \rho$ it has been shown that G/R_γ possesses an $\mathfrak{F}$-projector E_γ/R_γ such that

$$E_\gamma \leq E_\delta \text{, for } \delta \leq \gamma < \sigma,$$

and suppose also that we have a fixed Sylow basis $\underset{\sim}{S}$ of G, and that an $\mathfrak{F}$-normalizer D_γ of E_γ has been chosen such that

$$D_\delta \leq D_\gamma \text{, for } \delta \leq \gamma < \sigma,$$

and that D_γ is the $\mathfrak{F}$-normalizer of E_γ associated with $\underset{\sim}{S} \cap E_\gamma$.

Case (a). $\sigma - 1$ exists. Then we have that $E_{\sigma-1}/R_{\sigma-1}$ is an $\mathfrak{F}$-projector of $G/R_{\sigma-1}$. Hence $E_{\sigma-1}/R_\sigma \in (L\mathfrak{N})\mathfrak{F}$. Therefore, by Theorem 1.5.4, $E_{\sigma-1}/R_\sigma$ possesses an

$\mathfrak{F}$ -projector E_σ^*/R_σ.

By Lemma 1.4.5, E_σ^*/R_σ is $\mathfrak{F}$ -abnormal in $E_{\sigma-1}/R_\sigma$, and therefore E_σ^* is $\mathfrak{F}$ -abnormal in $E_{\sigma-1}$.

Now, let $\underset{\sim}{T}$ be a Sylow basis of $E_{\sigma-1}$ which reduces into E_σ^*, and let $D_{\sigma-1}^*$ and D_σ^* be the $\mathfrak{F}$ -normalizers of $E_{\sigma-1}$ associated with $\underset{\sim}{T}$ and E_σ^* associated with $\underset{\sim}{T} \cap E_\sigma^*$ respectively. Then, by Theorem 1.4.10. and Lemma 1.4.9,

$$D_{\sigma-1}^* \leqslant D_\sigma^* .$$

By Theorem 1.3.7, there exists $\alpha \in A(E_{\sigma-1})$ such that $(D_{\sigma-1}^*)^\alpha = D_{\sigma-1}$.

Let $E_\sigma = (E_\sigma^*)^\alpha$ and let $D_\sigma = (D_\sigma^*)^\alpha$. Then, D_σ is an $\mathfrak{F}$ -normalizer of E_σ, $D_{\sigma-1} \leqslant D_\sigma$, and E_σ/R_σ is an $\mathfrak{F}$ -projector of $E_{\sigma-1}/R_\sigma$.

It remains to show that $\underset{\sim}{S}$ reduces into E_σ. By Theorem 1.3.7, $(\underset{\sim}{T})^\alpha = \underset{\sim}{S} \cap E_{\sigma-1}$. Since $\underset{\sim}{T}$ reduces into E_σ^*, it follows that $(\underset{\sim}{T})^\alpha$ reduces into $(E_\sigma^*)^\alpha$, i.e.

$$\underset{\sim}{S} \text{ reduces into } E_\sigma .$$

Hence, by Lemma 1.5.5, E_σ/R_σ is an $\mathfrak{F}$ -projector of G/R_σ.

<u>Case (b)</u>. σ is a limit ordinal. Let $E_\sigma = \bigcap_{\gamma<\sigma} E_\gamma$. Since D_γ is an $\mathfrak{F}$ -normalizer of E_γ, $D_\gamma R_\gamma = E_\gamma$, for all $\gamma < \sigma$. But $D_\delta \leqslant D_\gamma \leqslant E_\gamma$, for all $\delta \leqslant \gamma < \sigma$.

Therefore

$$D_\gamma \leqslant E_\sigma, \text{ for all } \gamma < \sigma .$$

$$\text{Hence } E_\sigma R_\gamma = E_\gamma, \text{ for all } \gamma < \sigma.$$

We shall show that E_σ/R_σ is an $\mathfrak{F}$ -projector of G/R_σ. Suppose that $E_\sigma \leqslant U \leqslant G$. Clearly it is sufficient to show that E_σ contains an $\mathfrak{F}$ -normalizer of U.

So, let $\underset{\sim}{T}$ be a Sylow basis of E_σ, and extend it to a Sylow basis $\underset{\sim}{S}^*$ of U. Let D be the $\mathfrak{F}$ -normalizer of U associated with $\underset{\sim}{S}^*$.

Now E_γ/R_γ is an $\mathfrak{F}$ -projector of UR_γ/R_γ, for all $\gamma < \sigma$. Therefore $(E_\gamma \cap U)/(U \cap R_\gamma)$ is an $\mathfrak{F}$ -projector of $U/(U \cap R_\gamma)$, for all $\gamma < \sigma$.

Hence $E_\sigma(U \cap R_\gamma)/(U \cap R_\gamma)$ is $\mathfrak{F}$ -abnormal in $U/(U \cap R_\gamma)$.

Now $\underset{\sim}{S}^*(U \cap R_\gamma)/(U \cap R_\gamma)$ reduces into $E_\sigma(U \cap R_\gamma)/(U \cap R_\gamma)$ and so, by Lemma 1.4.9, and Theorem 1.4.10, $E_\sigma(U \cap R_\gamma)/(U \cap R_\gamma)$ contains the $\mathfrak{F}$ -normalizer of $U/(U \cap R_\gamma)$ associated with $\underset{\sim}{S}^*(U \cap R_\gamma)/(U \cap R_\gamma)$; this is $D(U \cap R_\gamma)/(U \cap R_\gamma)$. Therefore $D \leqslant E_\sigma(U \cap R_\gamma)$, for all $\gamma < \sigma$.

Therefore

$$D \leqslant \bigcap_{\gamma<\sigma} E_\sigma(U \cap R_\gamma) = E_\sigma, \text{ as required.}$$

Hence E_σ/R_σ is an $\mathfrak{F}$ -projector of G/R_σ.

Now, by Lemma 1.5.3, $\underset{\sim}{S}$ reduces into E_σ. So we have that E_σ is an $\mathfrak{F}$ -abnormal complete $\mathfrak{F}$ -residual supplement of E_γ, for all $\gamma \leqslant \sigma$. Let D_σ be the $\mathfrak{F}$ -normalizer of E_σ associated with $\underset{\sim}{S} \cap E_\sigma$.

Then, by Theorem 1.4.10, $D_\gamma \leqslant D_\sigma$, for all $\gamma < \sigma$.

Therefore

$$\bigcup_{\gamma<\sigma} D_\gamma \leqslant D_\sigma.$$

Hence, for all $\gamma \leqslant \sigma$, we have that E_γ/R_γ is an $\mathfrak{F}$ -projector of G/R_γ, D_γ is the $\mathfrak{F}$ -normalizer of E_γ associated with $\underset{\sim}{S} \cap E_\gamma$, and

$$E_\sigma \leqslant E_\gamma, \text{ and } D_\gamma \leqslant D_\sigma.$$

Hence, by transfinite induction, G possesses $\mathfrak{F}$ -projectors.

$\square$

Theorem 1.5.7. Let G be a $\mathcal{K}$-group, then the $\mathfrak{F}$-projectors of G form an A(G)-transitive class of subgroups.

Proof. Let E and E* be two $\mathfrak{F}$-projectors of G, and let

$$G = R_0 \geq R_1 \geq \dots \geq R_\sigma \geq \dots \geq R_\rho = 1$$

be the lower locally nilpotent series of G.

We shall prove by induction on σ, that there exists an automorphism of $A(G/R_\sigma)$ which maps ER_σ/R_σ onto E^*R_σ/R_σ.

If $\sigma = 0$, then the above is trivial.

If $\sigma = 1$, then the result follows by Theorems 1.5.4, and 1.3.7.

Assume that the above has been proved for all $\gamma < \sigma = \rho$.

Case (a). $\sigma - 1$ exists. Then there exists $\bar{\theta} \in A(G/R_{\sigma-1})$ such that

$$\bar{\theta}(ER_{\sigma-1}/R_{\sigma-1}) = E^*R_{\sigma-1}/R_{\sigma-1}.$$

Let $\theta \in A(G)$ be such that θ lifts $\bar{\theta}$. Then

$$\theta(E)R_{\sigma-1} = E^*R_{\sigma-1} \in (L\mathfrak{N})\mathfrak{F}.$$

Now, by Theorem 1.5.4, there exists $\bar{\varphi} \in A(E^*R_{\sigma-1})$ such that

$$\bar{\varphi}\theta(E) = E^* .$$

Let $\varphi \in A(G)$ be such that φ lifts $\bar{\varphi}$. Then $\varphi\theta(E) = E^*$, and $\varphi\theta \in A(G)$.

Case (b). σ is a limit ordinal. Then we may assume that for all $\gamma < \sigma$, there exist $\theta_\gamma \in A(G)$ such that

$$\theta_\gamma(ER_\gamma) = E^*R_\gamma .$$

Therefore, by definition of A(G), there exists $\theta \in A(G)$

such that $\theta\left(E(\bigcap_{\gamma<\sigma} R_\gamma) \right) = E^*(\bigcap_{\gamma<\sigma} R_\gamma)$, i.e.

$$\theta(E) = E^*.$$

$\square$

Corollary 1.5.8. If N is a normal subgroup of a $\mathfrak{K}$-group G, then the $\mathfrak{F}$-projectors of G/N are precisely the EN/N where E runs over the $\mathfrak{F}$-projectors of G.

The following theorem gives us more information about the relationship between $\mathfrak{F}$-normalizers and $\mathfrak{F}$-projectors of a $\mathfrak{K}$-group in the special case of $(L\mathfrak{N})^2\mathfrak{F}$-groups.

Theorem 1.5.9. Let G be a $\mathfrak{K} \cap (L\mathfrak{N})^2\mathfrak{F}$-group, then each $\mathfrak{F}$-normalizer of G is contained in a unique $\mathfrak{F}$-projector of G.

Proof. Let R be the locally nilpotent radical of G, and let D be the $\mathfrak{F}$-normalizer of G associated with the Sylow basis $\underset{\sim}{S}$ of G.

Then, if E is an $\mathfrak{F}$-projector of G containing D, DR/R is the $\mathfrak{F}$-normalizer of G/R associated with $\underset{\sim}{S}R/R$, and ER/R is an $\mathfrak{F}$-projector of G/R with

$$DR/R \leq ER/R.$$

Since G/R is an $(L\mathfrak{N})\mathfrak{F}$-group, it follows from Theorem 1.5.4. that

$$DR/R = ER/R, \text{ i.e. } DR = ER.$$

Therefore

$$E = D(E \cap R).$$

Now, by Lemma 1.1.8, $\underset{\sim}{S}$ reduces into $ER = DR$, since $\underset{\sim}{S}$ reduces into both D and R. Let $\underset{\sim}{T} = \underset{\sim}{S} \cap DR$.

Then $\underset{\sim}{T} \cap E$ extends to a unique Sylow basis of DR, namely $\underset{\sim}{T}$.

By Theorem 1.3.9, E is a subgroup of the $\mathfrak{F}$-normalizer of DR associated with $\underset{\sim}{T}$.

Hence E coincides with that $\mathfrak{F}$-normalizer, and so is uniquely determined by D.

$\square$

We now prove a result analogous to Theorem 1.3.9.

<u>Theorem 1.5.10</u>. Let $G = RH$ be a $\mathfrak{K}$-group, where R is a normal locally nilpotent subgroup of G. Then every $\mathfrak{F}$-projector of H has the form $E \cap H$ for some $\mathfrak{F}$-projector E of G.

<u>Proof</u>. Let E^*/R be an $\mathfrak{F}$-projector of G/R. Then E^* is an $(L\pi)\mathfrak{F}$-group, and so, by Theorem 1.5.4, any $\mathfrak{F}$-normalizer of E^* is an $\mathfrak{F}$-projector of E^*. By Lemma 1.5.5, any such $\mathfrak{F}$-normalizer will be an $\mathfrak{F}$-projector of G.

Now, $E^* = R(E^* \cap H)$, and so by Theorem 1.3.9, there exists an $\mathfrak{F}$-normalizer E of E^* such that

$$E^* \cap E \cap H = E \cap H$$

is an $\mathfrak{F}$-normalizer of $E^* \cap H$. By the above E is an $\mathfrak{F}$-projector of G.

Now, $G/R \cong H/(R \cap H)$, therefore $(E^* \cap H)/(R \cap H)$ is an $\mathfrak{F}$-projector of $H/(R \cap H)$.

By Lemma 1.5.5, since $E \cap H$ is an $\mathfrak{F}$-normalizer of $E^* \cap H$ and $E^* \cap H$ is an $(L\Pi)\mathfrak{F}$-group, $E \cap H$ is an $\mathfrak{F}$-projector of H.

The result follows by A(H)-transitivity.

□

<u>Theorem 1.5.11</u>. Let G be a $\mathfrak{K}$-group, $\underset{\sim}{S}$ a Sylow basis of G, and let D be the $\mathfrak{F}$-normalizer of G associated with $\underset{\sim}{S}$. Then there exists an $\mathfrak{F}$-projector E of G such that $D \leq E$ and $\underset{\sim}{S}$ reduces into E. Furthermore, each $\mathfrak{F}$-projector E^* of G contains the $\mathfrak{F}$-normalizer of G associated with some Sylow basis of G which reduces into E^*.

<u>Proof</u>. Let E_1 be an $\mathfrak{F}$-projector of G, and let $\underset{\sim}{T}$ be a Sylow basis of G which reduces into E_1.

Then there exists $\alpha \in A(G)$ such that $S = T^{\alpha}$. Let $E = E_1^{\alpha}$, then $\underset{\sim}{S}$ reduces into E and, by Theorem 1.4.10, it follows that $D \leq E$.

The last statement also follows immediately from Theorem 1.4.10.

□

1.6. $\mathcal{K}$ -GROUPS WITH ABELIAN SYLOW p -SUBGROUPS.

We define a group $G \in \mathcal{K}_{A,\pi}$ if and only if G is a $\mathcal{K}$ -group and has abelian Sylow p -subgroups for all $p \in \pi$. We say that $G \in \mathcal{K}_A$ if and only if $G \in \mathcal{K}_{A,\pi}$ for all sets of primes π.

Let H be a subgroup of a $\mathcal{K}$ -group G, and let S be a Sylow p -subgroup of H. We say that H is <u>p -normally embedded</u> in G, if S is a Sylow p -subgroup of N, for some normal subgroup N of G.

Clearly H is p -normally embedded in G if and only if S is a Sylow p -subgroup of S^G, for all Sylow p -subgroups S of H.

Let H be a subgroup of a $\mathcal{K}$ -group G, H is said to be <u>A -pronormal</u> in G if, given $\alpha \in A(G)$, there exists $\beta \in A(\langle H, H^\alpha \rangle)$ such that $H^\alpha = H^\beta$.

Let H be a subgroup of a $\mathcal{K}$ -group G, define

$$AN_G(H) = \{ \alpha \in A(G) \; ; \; H^\alpha = H \}$$

to be the <u>A -normalizer of H in G</u>.

The following result gives us a criterion for a subgroup H of a $\mathcal{K}$ -group G to be A -pronormal in G.

<u>Lemma 1.6.1</u>. Let H be a subgroup of a $\mathcal{K}$ -group G and let $\underset{\sim}{S}$ be a Sylow basis of G which reduces into H. Then H is A -pronormal in G if, whenever $\alpha \in A(G)$ and $\underset{\sim}{S}^\alpha$

reduces into H, then $\alpha \in AN_G(H)$.

Proof. Let $\alpha \in A(G)$, and let $\underset{\sim}{S}$ be a Sylow basis of G which reduces into both H and $< H, H^{\alpha} >$. Then there exists $\bar{\beta} \in A(< H, H^{\alpha} >)$ such that $(\underset{\sim}{S} \cap < H, H^{\alpha} >)^{\bar{\beta}}$ reduces into H^{α}. Lift $\bar{\beta}$ to $\beta \in A(G)$. Then both $\underset{\sim}{S}$ and $\underset{\sim}{S}^{\beta\alpha^{-1}}$ reduce into H.

Therefore, by hypothesis, $\beta\alpha^{-1} \in AN_G(H)$, i.e. $H^{\beta} = H^{\alpha}$.

But $H^{\beta} = H^{\bar{\beta}}$, hence H is A -pronormal in G.

□

Lemma 1.6.2. Let G be a $\mathcal{K}_{A,p}$ -group, then the p -length $l_p(G)$ of G is ≤ 1. In particular, $\mathcal{K}_A$ -groups have $l_p \leq 1$ for all primes p.

Proof. Let S be a Sylow p -subgroup of G, and let H/K be any p -chief factor of G, then

$$H/K \leq SK/K.$$

Now, S is an abelian group, and so S centralizes H/K.

Therefore, by Lemma 1.2.6,

$$S \leq O_{p'p}(G).$$

Therefore $G/O_{p'p}(G)$ is a p'-group.

Hence $l_p(G) \leq 1$.

□

Lemma 1.6.3. Let H be a subgroup of a $\mathcal{K}$ -group G, and suppose that H is p -normally embedded in G, for all primes p. Then H is A -pronormal in G.

<u>Proof</u>. Let $\underset{\sim}{S}$ be a Sylow basis of G, and suppose that $\underset{\sim}{S}$ and $\underset{\sim}{S}^{\alpha}$ both reduce into H, for some $\alpha \in A(G)$.

Then, $\underset{\sim}{S} \cap H$ and $\underset{\sim}{S}^{\alpha} \cap H$ are Sylow bases of H. Hence there exists $\beta \in A(H)$ such that

$$S_p \cap H = S_p^{\alpha\beta} \cap H = H_p \text{ say,}$$

for all p, where $\underset{\sim}{S} = \{ S_p \}$.

Now, H is p -normally embedded in G, and so there exists a normal subgroup N of G such that H_p is a Sylow p -subgroup of N.

But $H_p \leq S_p \cap S_p^{\alpha\beta}$, so

$$H_p = S_p \cap N = S_p^{\alpha\beta} \cap N.$$

Therefore

$$H_p^{\alpha\beta} = (S_p \cap N)^{\alpha\beta} = S_p^{\alpha\beta} \cap N = H_p.$$

Therefore $\alpha\beta \in AN_G(H)$, and so $\alpha \in AN_G(H)$, since $\beta \in A(H)$.

Therefore, by Lemma 1.6.1, H is A -pronormal in G.

$\square$

<u>Lemma 1.6.4</u>. Let G be a $\mathcal{K}_{A,p}$ -group, and let H be a subgroup of G such that H contains a Sylow p'-subgroup S of G. Then H is p -normally embedded in G.

<u>Proof</u>. By Lemma 1.6.2, $l_p(G) \leq 1$. Therefore

$$G = O_{p'pp'}(G).$$

<u>Case (a)</u>. $O_{p'}(G) = 1$. Then G has a unique abelian Sylow p -subgroup P, say.

Since G = PS, it follows that G = HP.

Therefore $H \cap P$ is a normal subgroup of G.

But, P contains the unique Sylow p -subgroup of H.

Therefore $H \cap P$ is a Sylow p-subgroup of H.

Case (b). $O_{p'}(G) \neq 1$. Let H_p be a Sylow p-subgroup of H, then $H_pO_{p'}(G)/O_{p'}(G)$ is a Sylow p-subgroup of $HO_{p'}(G)/O_{p'}(G)$.

Therefore, by case (a), $H_pO_{p'}(G)/O_{p'}(G)$ is a normal subgroup of $G/O_{p'}(G)$.

Hence $H_pO_{p'}(G)$ is a normal subgroup of G.
Any subgroup of $H_pO_{p'}(G)$ properly containing H_p contains a non - trivial p'-element.

Therefore H_p is a Sylow p-subgroup of $H_pO_{p'}(G)$.
Therefore H is p-normally embedded in G.

□

We shall show later that in $\mathcal{K}_A$-groups the $\mathfrak{F}$-normalizers complement the $\mathfrak{F}$-residual. To do this we require the following generalization of a result of Taunt 20.

Lemma 1.6.5. Let G be a $\mathcal{K}$-group, and suppose that the Sylow p-subgroup S of G is abelian. Then

$$S \cap G' \cap Z(G) = 1.$$

Proof. Suppose that there exists $1 \neq x \in S \cap G' \cap Z(G)$. Then

$$x = [y_1, z_1] \ldots\ldots [y_n, z_n], \text{ for some } y_i, z_i \in G,$$
$1 \leqslant i \leqslant n$.

Let $G_1 = \langle y_i, z_i \, ; \, 1 \leqslant i \leqslant n \rangle$. Then G_1 is a finite group, and if S_1 is a Sylow p-subgroup of G_1 containing $S \cap G_1$, then

$$S_1 \cap G_1' \cap Z(G_1) = 1, \text{ by the finite case}$$

mentioned above.

But $1 \neq x \in S_1 \cap G_1' \cap Z(G_1)$, which is a contradiction.

□

<u>Corollary 1.6.6</u>. If G is a $\mathcal{K}_A$ -group, then $G' \cap Z(G) = 1$.

For the remainder of this section let f be a $\mathcal{K}$ -preformation function defined on a set of primes π.

<u>Lemma 1.6.7</u>. Let G be a $\mathcal{K}_{A,p}$ -group, let S be a Sylow p'-subgroup of G, and let $C_p(G)$ be the $f(p)$ -centralizer of G, for some prime $p \in \pi$. Then $N_G(\ S \cap C_p(G)\ )$ is p -normally embedded in G.

<u>Proof</u>. Since $S \leqslant N_G(\ S \cap C_p(G)\ )$ the result follows immediately from Lemma 1.6.4.

□

<u>Corollary 1.6.8</u>. Let $p \in \pi$ and let G be a $\mathcal{K}_{A,p}$ -group. If D is an $\mathfrak{F}$ -normalizer of G, then D is p -normally embedded in G.

<u>Proof</u>. Suppose that D is the $\mathfrak{F}$ -normalizer of G associated with the Sylow basis $\underset{\sim}{S}$ of G. Then,

$$S_p \cap D = S_p \cap N_G(\ S_{p'} \cap C_p(G)\ )$$

is a Sylow p -subgroup of D and also of $N_G(\ S_{p'} \cap C_p(G)\ )$, where $\underset{\sim}{S} = \{\ S_p\ \}$.

The result is now immediate from Lemma 1.6.7.

□

We now establish the following generalization of [4] 3.5.

<u>Theorem 1.6.9</u>. Let G be a $\mathcal{K}_{A,\pi}$-group, and let D be an $\mathfrak{F}$-normalizer of G. Then D is p-normally embedded in G for all primes p, and hence is A-pronormal in G.

<u>Proof</u>. If $p \in \pi$, then D is p-normally embedded in G, by Corollary 1.6.8.

If $p \notin \pi$, then $D_p = 1$, since D is a π-group by definition. Hence D is trivially p-normally embedded in G.

Therefore D is p-normally embedded in G for all primes p.

Hence, by Lemma 1.6.3, D is A-pronormal in G.

$\square$

<u>Corollary 1.6.10</u>. The $\mathfrak{F}$-normalizers of $\mathcal{K}_A$-groups are A-pronormal.

The following corollary to Theorem 1.6.9. gives us a partial extension of Alperin's result [1] Theorem 2.

<u>Corollary 1.6.11</u>. If G is a $\mathcal{K}_{A,\pi}$-group, and D and D^* are $\mathfrak{F}$-normalizers of G contained in some $\mathfrak{F}$-projector E of G, then there exists $\alpha \in A(E)$ such that $D^{\alpha} = D^*$.

<u>Proof</u>. Now, by Theorem 1.3.7, there exists $\beta \in A(G)$ such that $D = D^{*\beta}$,

Now, by Theorem 1.6.9, D^* is A -pronormal in G, therefore there exists $\gamma \in A(< D^*, D^{*\beta} >)$ such that

$$D^{*\gamma} = D^{*\beta}.$$

Lift γ to $\alpha^{-1} \in A(E)$, then $D^{*\alpha^{-1}} = D^{*\gamma} = D$.

$$\text{Therefore } D^{\alpha} = D^*.$$

$\square$

<u>Remark</u>. Corollary 1.6.11. is false in general as Hawkes' example $\underset{\sim}{13}$ demonstrates.

We now discuss Chambers' characterization of $\mathfrak{F}$ -normalizers (of finite soluble A -groups) by the covering/avoiding property. We shall prove that a similar characterization holds for the $\mathfrak{F}$ -normalizers of $\mathcal{K}_{A,\pi}$-groups.

<u>Lemma 1.6.12</u>. Let G be a $\mathcal{K}_{A,p}$ -group with $p \in \pi$, and let H be a p -subgroup of G which avoids every $\mathfrak{F}$ -eccentric p -chief factor of G. Then $H \leq N_G(S \cap C_p(G))$, for all Sylow p'-subgroups S of G.

<u>Proof</u>. Let S be a Sylow p'-subgroup of G, and let $N = N_G(S \cap C_p(G))$, then $S \leq N$.

Let $P = O_{p'p}(G)$. Then G/P is a p'-group, by Lemma 1.6.2.

Therefore

$$G = PS = PN.$$

Now, $O_{p'}(G) \leq P \cap N \leq P$, and $P/O_{p'}(G)$ is an abelian group. Therefore $P \cap N$ is a normal subgroup of $PN = G$.

Let $C = P \cap N$.

By Lemma 1.2.2 and the definition of $C_p(G)$, N covers the $\mathfrak{F}$-central p -chief factors of G and avoids the $\mathfrak{F}$-eccentric ones.

Now, N avoids every chief factor of G between C and P, and so such factors are $\mathfrak{F}$-eccentric.

By hypothesis therefore H avoids every chief factor of G between C and P.

Let $x \to x^*$ be the natural epimorphism of $G \to G^*$, where $G^* = G/O_{p'}(G)$.

Then $H^* \leqslant P^*$, since H is a p -subgroup of G. Suppose that $H^* \nleqslant C^*$, and let $x^* \notin C^*$. Then $x^* \in P^*$ and $x^* \notin C^*$, so there is a chief factor U^*/V^* of G^* such that

$$x^* \in U^*,\ x^* \notin V^*, \text{ and } C^* \leqslant V^* < U^* \leqslant P^*.$$

Now H avoids U/V, so H^* avoids U^*/V^*.

Therefore $H^* \cap U^* = H^* \cap V^*$, which is a contradiction since $x^* \in H^* \cap U^*$ but $x^* \notin H^* \cap V^*$.

Therefore $H^* \leqslant C^*$, and hence $H \leqslant C \leqslant N$.

$\square$

<u>Lemma 1.6.13</u>. Let H be a subgroup of a $\mathcal{K}_{A,\pi}$-group G, and suppose that H avoids every $\mathfrak{F}$-eccentric chief factor of G. Then H is contained in some $\mathfrak{F}$-normalizer of G.

<u>Proof</u>. We first show that H is a π-group. For, suppose not.

Then there exists an element x of prime order q, say, which belongs to H such that $q \notin \pi$.

Let $\{ U_\sigma, V_\sigma ; \sigma \in \Omega \}$ be a chief series of G.

Then, there exists $\sigma \in \Omega$ such that $x \in U_\sigma$ and $x \notin V_\sigma$. Since $x \in H \cap U_\sigma$ and $x \notin H \cap V_\sigma$, H does not avoid U_σ/V_σ, and so by hypothesis U_σ/V_σ is $\mathfrak{F}$-central. Hence, if U_σ/V_σ is a p-chief factor, $p \in \pi$.

But xV_σ is a non - trivial element of order q in U_σ/V_σ, so $q = p \in \pi$, which is a contradiction.

Hence H is a π-group.

Let $\underset{\sim}{S}$ be a Sylow basis of G which reduces into H. If $p \in \pi$, then $H_p = H \cap S_p$ is a p-subgroup of H which avoids every $\mathfrak{F}$-eccentric p-chief factor of G.

So, by Lemma 1.6.12,

$$H_p \leqslant N_G(\, S_{p'} \cap C_p(G)\,).$$

Since

$$H_{p'} = H \cap S_{p'} \leqslant S_{p'} \leqslant N_G(\, S_{p'} \cap C_p(G)\,),$$

it follows that

$$H \leqslant N_G(\, S_{p'} \cap C_p(G)\,), \quad \text{for all } p \in \pi.$$

But H is a π-group, so $H = H \cap S_\pi$.

Therefore

$$H \leqslant S_\pi \cap \bigcap_{p \in \pi} N_G(\, S_{p'} \cap C_p(G)\,),$$ which is the $\mathfrak{F}$-normalizer of G associated with $\underset{\sim}{S}$.

$\square$

<u>Lemma 1.6.14</u>. Let G be a $\mathcal{K}_{A,\pi}$-group, and let D be an $\mathfrak{F}$-normalizer of G. Then,

(i) every chief factor of G below the $\mathfrak{F}$-residual F of G is $\mathfrak{F}$-eccentric,

(ii) D complements F in G.

Proof. (i) Suppose, for a contradiction, that the assertion is false. Then, there exists an $\mathfrak{F}$-central chief factor H/K of G below the $\mathfrak{F}$-residual F of G.

Then H/K is an $\mathfrak{F}$-central minimal normal subgroup of G/K contained in the $\mathfrak{F}$-residual F/K of G/K.

Without loss of generality we may assume that $K = 1$. Now $F \leq C_G(H)$, since f is integrated, and H is $\mathfrak{F}$-central.

Therefore

$$H \leq Z(F).$$

If H is a p-group, then $p \in \pi$ since H is $\mathfrak{F}$-central in G, and F has abelian Sylow p-subgroups by hypothesis.

Therefore $H \cap F' = 1$, by Lemma 1.6.5.

So, the $\mathfrak{F}$-residual F/F' of G/F' is an abelian group, and so G/F' is an $\mathfrak{A}\mathfrak{F}$-group.

Therefore, by Theorem 1.3.11, the $\mathfrak{F}$-normalizers of G/F' complement F/F' in G/F'.

Therefore, by Theorem 1.3.7, every chief factor of G/F' below F/F' is $\mathfrak{F}$-eccentric.

But $H \cap F' = 1$, so H is G-isomorphic to HF'/F'. Therefore HF'/F' is an $\mathfrak{F}$-central chief factor of G/F' below F/F', which is a contradiction.

(ii) Suppose that there exists $1 \neq x \in D \cap F$. Then, by taking a chief series of G passing through F, we obtain a chief factor X/Y of G below F such that $x \in X$, and $x \notin Y$.

Since $x \in D \cap X$ and $x \notin D \cap Y$ the chief factor X/Y is $\mathfrak{F}$-central, by Theorem 1.3.7.

But this is a contradiction to (i).

Hence $D \cap F = 1$.

$\square$

The following result will be useful in proving our main theorem.

<u>Lemma 1.6.15</u>. Let G be an SD -group, and suppose that H is a subgroup of G which covers every chief factor of G. Then, if either (i) G is soluble,

or (ii) H is a distributive subgroup of G,

then H = G.

<u>Proof</u>. (i) This is Lemma 3.2.7. of 8.

(ii) If N is a normal subgroup of G, then clearly HN/N covers every chief factor of G/N.

Let

$$G = D_0 \geq D_1 \geq \dots \geq D_\sigma \geq \dots \geq D_\rho = 1$$

be the derived series of G.

We shall prove, by induction on σ, that $HD_\sigma = G$.

If $\sigma = 0$, then the above is trivial.

If $\sigma = 1$, then it follows by (i).

Assume that it has been proved for all $\sigma < \rho$.

<u>Case (a)</u>. $\rho - 1$ exists. Then $G = HD_{\rho-1}$.

Since $D_{\rho-1}$ is an abelian group, $H \cap D_{\rho-1}$ is a normal subgroup of G. So we may take a chief series of G passing through $H \cap D_{\rho-1}$ and $D_{\rho-1}$.

If $H \cap D_{\rho-1} < D_{\rho-1}$, then H does not cover any chief factor between $H \cap D_{\rho-1}$ and $D_{\rho-1}$, which contradicts our hypothesis.

Therefore $H \cap D_{\rho-1} = D_{\rho-1}$ and hence H = G.

<u>Case (b)</u>. ρ is a limit ordinal. Then $G = HD_\sigma$, for all $\sigma < \rho$.

Therefore

$$G = \bigcap_{\sigma<\rho} HD_{\sigma} = H(\bigcap_{\sigma<\rho} D_{\sigma}) = H. \qquad \square$$

<u>Remark</u>. Lemma 1.6.15 fails to hold if either of the conditions are removed. For Hartley 12 gives an example of a locally finite p -group P with a proper subgroup Q which covers every chief factor of P.

We can now prove our generalization of 4 3.6.

<u>Theorem 1.6.16</u>. Let H be a subgroup of a $\mathcal{K}_{A,\pi}$-group G. Then,

(i) if $\mathfrak{F} \leq PL\Pi$, H is an $\mathfrak{F}$ -normalizer of G if and only if H covers every $\mathfrak{F}$ -central chief factor of G and avoids every $\mathfrak{F}$ -eccentric one,

(ii) if $\mathfrak{F}$ is an arbitrary saturated formation, then H is an $\mathfrak{F}$ -normalizer of G if and only if H is a distributive subgroup of G which covers every $\mathfrak{F}$ -central chief factor of G and avoids every $\mathfrak{F}$ -eccentric one.

<u>Proof</u>. (i) By Theorem 1.3.7, $\mathfrak{F}$ -normalizers possess the required property, so we need only show that a subgroup H with the given property is an $\mathfrak{F}$ -normalizer of G.

Now, let F be the $\mathfrak{F}$ -residual of G, then

$$G/F \in \mathfrak{F} \leq PL\Pi,$$

so in particular, G/F is a π -group and therefore has abelian Sylow p -subgroups for all $p \in \pi$, by hypothesis.

Clearly HF/F covers every chief factor of G/F, since all such factors are $\mathfrak{F}$ -central.

Therefore, by Lemma 1.6.15, G = HF.

Now, by Lemma 1.6.13, H is contained in some $\mathfrak{F}$ -normalizer D of G, and D complements F in G, by Lemma 1.6.14.

Therefore

$$D = D \cap HF = H(D \cap F) = H.$$

(ii) By Theorem 1.3.7 and Lemma 1.5.1, $\mathfrak{F}$ -normalizers have the required property, so let H be a subgroup of G with the given property.

As in (i), HF/F covers every chief factor of G/F, since all such factors are $\mathfrak{F}$ -central.

Therefore, by Lemma 1.6.15, G = HF.

The same argument as that used in (i) shows that H is an $\mathfrak{F}$ -normalizer of G.

$\square$

PART TWO

$\mathcal{V}$-GROUPS

2.1. SYLOW STRUCTURE IN $\mathfrak{U}$-GROUPS.

In this section we shall consider a particular subclass $\mathfrak{U}$ of $(\mathfrak{W}$, Linn), where if G is an arbitrary group, then Linn(G) denotes the group of all locally inner automorphisms of G. We shall show that even without the pronormality axiom groups can possess a 'good' Sylow structure; however these groups will not in general possess $\mathfrak{F}$-projectors: see A1.

We shall say that a group $G \in \bar{\mathfrak{U}}$ if and only if G possesses a local system Σ of finite soluble subgroups such that the locally nilpotent residual R of G is contained in the normalizer of the local system, i.e.

$$R \leq N_G(\Sigma) = \bigcap_{F \in \Sigma} N_G(F) \ .$$

It is clear that the above class is QS-closed, and that if R is the locally nilpotent residual of a $\bar{\mathfrak{U}}$-group G, then R is an FC-group. Hence $\bar{\mathfrak{U}} \leq (FC)(L\mathfrak{N})$. It is also clear that F ser G, for all $F \in \Sigma$.

Throughout, when considering $\bar{\mathfrak{U}}$-groups, we shall assume that Σ denotes a local system which defines the group in question, i.e. satisfies the above definition.

<u>Lemma 2.1.1</u>. Let G be a $\bar{\mathfrak{U}}$-group, and let S be a subgroup of G. Then, S is a Sylow π-subgroup of G if and only if $S \cap F$ is a Sylow π-subgroup of F, for all $F \in \Sigma$.

<u>Proof</u>. Let us first assume that S is a Sylow π-subgroup of G.

Let $\Sigma = \{ F_\lambda \; ; \; \lambda \in \Lambda \}$. We shall define a partial ordering on Λ by, $\alpha \leq \beta$ whenever $F_\alpha \leq F_\beta$.

Let $S_\lambda = S \cap F_\lambda$, for all $\lambda \in \Lambda$, and let

$A_\lambda = \{$ Sylow π-subgroups of F_λ containing $S_\lambda \}$.

Then A_λ is finite and non - empty for all $\lambda \in \Lambda$.

For $\alpha \leq \beta$, define $p_{\beta\alpha} : A_\beta \to A_\alpha$ by

$$p_{\beta\alpha}(T_\beta) = T_\beta \cap F_\alpha \text{, for } T_\beta \in A_\beta .$$

This is well-defined, since, if R is the locally nilpotent residual of G, F_α is a normal subgroup of $F_\alpha(F_\beta \cap R)$, and

$$\begin{aligned} F_\alpha(F_\beta \cap R)/(F_\beta \cap R) &\leq F_\beta/(F_\beta \cap R) \\ &\cong F_\beta R/R \in \Pi^*. \end{aligned}$$

Therefore $F_\alpha(F_\beta \cap R)$ is subnormal in F_β, and so F_α is subnormal in F_β.

Hence $F_\alpha \cap T_\beta$ is a Sylow π-subgroup of F_α.

Clearly the sets $\{ A_\lambda \; ; \; \lambda \in \Lambda \}$ and maps $\{ p_{\beta\alpha} \}$ form a projection set.

Hence there exist $T_\lambda \in A_\lambda$, one for each $\lambda \in \Lambda$, such that if $\alpha \leq \beta$, then $p_{\beta\alpha}(T_\beta) = T_\alpha$.

Then $T = \bigcup_{\lambda \in \Lambda} T_\lambda$ is a Sylow π-subgroup of G which contains S and reduces into F, for all $F \in \Sigma$.

Therefore $S = T$.

Conversely, let $x \in S$, then there exists $F \in \Sigma$ such that $x \in F$.

Therefore $x \in S \cap F$ and $S \cap F$ is a Sylow π-subgroup of F.

Therefore $< x >$, and hence S, is a π-subgroup of G. Suppose that S is contained in a Sylow π-subgroup T of G.

Let $t \in T$, then there exists $F \in \Sigma$ such that $t \in F$.

Therefore $t \in T \cap F \geq S \cap F$, but $S \cap F$ is a Sylow π-subgroup of F.

Therefore

$$t \in T \cap F = S \cap F.$$

Hence $S = T$ and is a Sylow π-subgroup of G.

$\square$

<u>Theorem 2.1.2.</u> Let G be a $\overline{\mathfrak{U}}$-group, and let S and T be Sylow π-subgroups of G. Then S and T are locally conjugate in G.

<u>Proof.</u> Let $\Sigma = \{ F_\lambda \,;\, \lambda \in \Lambda \}$ and define a partial ordering on Λ by $\alpha \leq \beta$ whenever $F_\alpha \leq F_\beta$. Let R be the locally nilpotent residual of G.

Now, suppose that $F_\alpha, F_\beta \in \Sigma$ and $F_\alpha \leq F_\beta$. Then

$$F_\beta R/R \cong F_\beta/(F_\beta \cap R) \text{ is a nilpotent group.}$$

Therefore, since $S \cap F_\beta$ and $T \cap F_\beta$ are both Sylow π-subgroups of F_β by Lemma 2.1.1, we have

$$(S \cap F_\beta)(F_\beta \cap R) = (T \cap F_\beta)(F_\beta \cap R).$$

Therefore there exists $h_\beta \in F_\beta \cap R$ such that

$$(S \cap F_\beta)^{h_\beta} = T \cap F_\beta .$$

But $h_\beta \in N_G(F_\alpha)$, so $(S \cap F_\alpha)^{h_\beta} \leq T \cap F_\alpha$.

Therefore

$$(S \cap F_\alpha)^{h_\beta} = T \cap F_\alpha .$$

Let φ_β be the automorphism of F_β which is induced by h_β. Let

$A_\lambda = \{$ automorphisms of F_λ which are induced by conjugation by an element of R and which map $S \cap F_\lambda$ onto $T \cap F_\lambda$ $\}$.

Then A_λ is finite and non - empty for all $\lambda \in \Lambda$.

If $F_\alpha \leq F_\beta$, define $p_{\beta\alpha} : A_\beta \to A_\alpha$ by

$$p_{\beta\alpha}(\varphi_\beta) = \varphi_\beta|_{F_\alpha}.$$

Clearly the sets $\{ A_\lambda ; \lambda \in \Lambda \}$ and maps $\{ p_{\beta\alpha} \}$ form a projection set.

Hence there exist $\varphi_\alpha \in A_\alpha$, one for each $\alpha \in \Lambda$, such that if $\alpha \leq \beta$, then $p_{\beta\alpha}(\varphi_\beta) = \varphi_\alpha$.

Then the $\{ \varphi_\lambda ; \lambda \in \Lambda \}$ define a locally inner automorphism of G which maps S onto T.

Hence S and T are locally conjugate in G.

$\square$

<u>Lemma 2.1.3</u>. Let K be a normal subgroup of a $\overline{\mathfrak{U}}$-group G and let θ be an automorphism of G/K which is induced locally by conjugation via the locally nilpotent residual R of G. Then there exists a locally inner automorphism φ of G effected locally by conjugation by elements of R such that φ induces θ in G/K.

<u>Proof</u>. Let $\Sigma = \{ F_\lambda ; \lambda \in \Lambda \}$, and let F_α and $F_\beta \in \Sigma$ with $F_\alpha \leq F_\beta$. Consider $F_\beta K/K$. Then

$$\theta(F_\beta K/K) = F_\beta^g K/K, \text{ for some } g \in R.$$

Let φ_β be the automorphism of F_β which is induced by g. Then $\varphi_\beta|_{F_\alpha}$ is an automorphism of F_α, since $g \in N_G(F_\alpha)$.

Let

$A_\lambda = \{$ automorphisms of F_λ which are induced by conjugation by an element of R and which agree with θ on $F_\lambda K/K$ $\}$.

Then A_λ is finite and non - empty for all $\lambda \in \Lambda$.

If $F_\alpha \leqslant F_\beta$, define $p_{\beta\alpha} : A_\beta \to A_\alpha$ by

$$p_{\beta\alpha}(\varphi_\beta) = \varphi_\beta|_{F_\alpha}.$$

Clearly the sets $\{ A_\lambda ; \lambda \in \Lambda \}$ and maps $\{ p_{\beta\alpha} \}$ form a projection set.

Hence, by the usual argument, we obtain a locally inner automorphism φ of G with the required property.

□

Lemma 2.1.4. Let K be a subgroup of a $\bar{\mho}$-group G, and let θ be an automorphism of K which is induced locally by conjugation via the locally nilpotent residual R of G. Then there exists a locally inner automorphism φ of G effected locally by conjugation by elements of R such that $\varphi|_K$ agrees with θ.

Proof. Let $\Sigma = \{ F_\lambda ; \lambda \in \Lambda \}$ and let F_α and $F_\beta \in \Sigma$ with $F_\alpha \leqslant F_\beta$. Consider $K \cap F_\beta$. Then

$$\theta(K \cap F_\beta) = (K \cap F_\beta)^g, \text{ for some } g \in R.$$

Let φ_β be the automorphism of F_β induced by g. Then $\varphi_\beta|_{F_\alpha}$ is an automorphism of F_α, since $g \in N_G(F_\alpha)$.

Let

$A_\lambda = \{$ automorphisms of F_λ which are induced by conjugation by an element of R and which agree with θ on $K \cap F_\lambda \}$.

Then A_λ is finite and non - empty for all $\lambda \in \Lambda$.

If $F_\alpha \leqslant F_\beta$ define $p_{\beta\alpha} : A_\beta \to A_\alpha$ by

$$p_{\beta\alpha}(\varphi_\beta) = \varphi_\beta|_{F_\alpha}.$$

Clearly the sets $\{ A_\lambda ; \lambda \in \Lambda \}$ and maps $\{ p_{\beta\alpha} \}$ form a projection set.

Hence, by the usual argument, we obtain a locally inner automorphism φ of G with the required property.

$\square$

Lemma 2.1.5. Let G be a $\bar{\mathfrak{U}}$-group, then the set $\underset{\sim}{S} = \{ S_p \}$ is a Sylow basis of G if and only if $\underset{\sim}{S} \cap F$ is a Sylow basis of F, for all $F \in \Sigma$.

Proof. First let us assume that $\underset{\sim}{S}$ is a Sylow basis of G. Let $F \in \Sigma$, then, by Lemma 2.1.1,

$S_p \cap F$ is a Sylow p-subgroup of F, for all p.

Let π be a set of primes, and let

$$S(\pi) = \langle S_p \; ; \; p \in \pi \rangle.$$

Since $\underset{\sim}{S}$ is a Sylow basis of G, $S(\pi)$ is a π-group.

Hence the same is true of $\langle S_p \cap F \; ; \; p \in \pi \rangle$.

Therefore $\underset{\sim}{S} \cap F$ is a Sylow basis of F, for all $F \in \Sigma$.

Conversely, by Lemma 2.1.1, S_p is a Sylow p-subgroup of G, for all p.

Let $x \in S(\pi)$, then there exist $x_1, \ldots, x_n$ with $x_i \in S_{p_i}$, $1 \leq i \leq n$, such that

$$x \in \langle x_1, \ldots, x_n \rangle = K, \text{ say.}$$

Then there exists $F \in \Sigma$ such that $K \leq F$.

Therefore, since $x \in F$, x is a π-element, and hence $S(\pi)$ is a π-group.

Therefore $\underset{\sim}{S}$ is a Sylow basis of G.

$\square$

Theorem 2.1.6. Let G be a $\bar{\mathfrak{U}}$-group, then G possesses Sylow bases.

<u>Proof</u>. Let $\Sigma = \{ F_\lambda ; \lambda \in \Lambda \}$, and let F_α and $F_\beta \in \Sigma$ with $F_\alpha \leq F_\beta$.

Since F_β is a finite soluble group it possesses Sylow bases.

If $\underset{\sim}{S}$ is a Sylow basis of F_β, then $\underset{\sim}{S} \cap F_\alpha$ is a Sylow basis of F_α, since F_α is subnormal in F_β.

Let $A_\lambda = \{$ Sylow bases of $F_\lambda \}$. Then A_λ is finite and non - empty for all $\lambda \in \Lambda$.

If $F_\alpha \leq F_\beta$, define $p_{\beta\alpha} : A_\beta \to A_\alpha$ by

$$p_{\beta\alpha}(\underset{\sim}{S}) = \underset{\sim}{S} \cap F_\alpha, \text{ where } \underset{\sim}{S} \in A_\beta .$$

Clearly the sets $\{ A_\lambda ; \lambda \in \Lambda \}$ and maps $\{ p_{\beta\alpha} \}$ form a projection set.

Hence there exist $\underset{\sim}{S}^\lambda \in A_\lambda$, one for each $\lambda \in \Lambda$, such that if $\alpha \leq \beta$ then $p_{\beta\alpha}(\underset{\sim}{S}^\beta) = \underset{\sim}{S}^\alpha$.

Let F_p^λ be the Sylow p -subgroup of F_λ in $\underset{\sim}{S}^\lambda$, for all $\lambda \in \Lambda$.

Then clearly

$$\underset{\sim}{S} = \{ G_p ; G_p = \bigcup_{\lambda \in \Lambda} F_p^\lambda \}$$

is a Sylow basis of G.

$\square$

<u>Lemma 2.1.7</u>. Let H be a subgroup of a $\bar{\mathfrak{U}}$ -group G, and let $\underset{\sim}{T}$ be a Sylow basis of H. Then there exists a Sylow basis $\underset{\sim}{S}$ of G such that $\underset{\sim}{S} \cap H = \underset{\sim}{T}$.

<u>Proof</u>. Let $\Sigma = \{ F_\lambda ; \lambda \in \Lambda \}$. Then it is clear that $\Sigma \cap H$ is a local system of H which defines H to be a $\bar{\mathfrak{U}}$ -group.

Let F_α and $F_\beta \in \Sigma$ and suppose that $F_\alpha \leq F_\beta$.

By Lemma 2.1.5, $\underset{\sim}{T} \cap H \cap F_\beta$ is a Sylow basis of $H \cap F_\beta$.

Hence, by the finite case, there exists a Sylow basis $\underset{\sim}{U}$, say, of F_β such that

$$\underset{\sim}{U} \cap H \cap F_\beta = \underset{\sim}{T} \cap H \cap F_\beta .$$

Now, since F_α is subnormal in F_β, it follows that $\underset{\sim}{U} \cap F_\alpha$ is a Sylow basis of F_α which extends $\underset{\sim}{T} \cap H \cap F_\alpha$.

Let

$$A_\lambda = \{ \text{ Sylow bases of } F_\lambda \text{ which extend } \underset{\sim}{T} \cap H \cap F_\lambda \}$$

Then A_λ is finite and non - empty for all $\lambda \in \Lambda$.

If $F_\alpha \leqslant F_\beta$, define $p_{\beta\alpha} : A_\beta \to A_\alpha$ by

$$p_{\beta\alpha}(\underset{\sim}{U}^\beta) = \underset{\sim}{U}^\beta \cap F_\alpha , \text{ where } \underset{\sim}{U}^\beta \in A_\beta .$$

Clearly the sets $\{ A_\lambda \ ; \lambda \in \Lambda \}$ and maps $\{ p_{\beta\alpha} \}$ form a projection set.

Hence there exist $\underset{\sim}{U}^\lambda \in A_\lambda$, one for each $\lambda \in \Lambda$, such that if $\alpha \leqslant \beta$, then $p_{\beta\alpha}(\underset{\sim}{U}^\beta) = \underset{\sim}{U}^\alpha$.

Let F_p^λ be the Sylow p -subgroup of F_λ in $\underset{\sim}{U}^\lambda$, for all $\lambda \in \Lambda$.

Then clearly

$$\underset{\sim}{S} = \{ G_p \ ; G_p = \bigcup_{\lambda \in \Lambda} F_p^\lambda \}$$

is a Sylow basis of G such that $\underset{\sim}{S} \cap H = \underset{\sim}{T}$.

$\square$

Before we prove the local conjugacy of Sylow bases of $\overline{U}$ -groups, we need to modify a result of P. Hall $\underset{\sim}{9}$.

<u>Lemma 2.1.8.</u> Let $\underset{\sim}{S}$ and $\underset{\sim}{T}$ be Sylow bases of a finite soluble group G. Then there exists an element g which belongs to the nilpotent residual of G such that $\underset{\sim}{S}^g = \underset{\sim}{T}$.

<u>Proof</u>. Let $\underset{\sim}{S}$ and $\underset{\sim}{T}$ be defined by Sylow p -complements

$$\underset{\sim}{S} = \{ S_1, \ldots , S_r \} \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$\underset{\sim}{T} = \{ T_1, \ldots , T_r \} \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

where S_i and T_i have index $p_i^{a_i}$, say, in G.

Suppose that (1) and (2) have k terms in common.

If $k = r$, then the result is trivial.

So, let $k < r$. Suppose that $S_i \neq T_i$, and let S_i^* be the Sylow p_i -subgroup of $\underset{\sim}{S}$.

Then $G = S_iS_i^*$. Also, $S_iR = T_iR$, where R is the nilpotent residual of G.

But $R = R_iR_i^*$, where $R_i = S_i \cap R$, and $R_i^* = S_i^* \cap R$.

Therefore

$$S_iR = S_iR_iR_i^* = S_iR_i^* = T_iR.$$

Therefore there exists $x \in R_i^*$ such that $S_i^x = T_i$.

Now,

$S_i^* = \bigcap_{j \neq i} S_j$, hence $x \in S_j$, for all $j \neq i$.

Therefore $S_j^x = S_j$, for all $j \neq i$.

Therefore $\underset{\sim}{S}^x$ and $\underset{\sim}{T}$ have $k + 1$ terms in common. Hence, by induction, there exists $y \in R$ such that

$$\underset{\sim}{S}^y = \underset{\sim}{T}.$$

$\square$

<u>Theorem 2.1.9</u>. Let G be a $\overline{\mathfrak{U}}$ -group, then any two Sylow bases of G are locally conjugate in G.

<u>Proof</u>. Let $\Sigma = \{ F_\lambda ; \lambda \in \Lambda \}$, and let F_α and $F_\beta \in \Sigma$ with $F_\alpha \leq F_\beta$.

Let $\underset{\sim}{S}$ and $\underset{\sim}{T}$ be Sylow bases of G, then, by Lemma 2.1.5, $\underset{\sim}{S} \cap F_\beta$ and $\underset{\sim}{T} \cap F_\beta$ are Sylow bases of F_β.

Therefore, by Lemma 2.1.8, there exists $g \in N \leq R$ such that

$$(\underset{\sim}{S} \cap F_\beta)^g = \underset{\sim}{T} \cap F_\beta \text{ , where } N \text{ is the nilpotent}$$

residual of F_β and R is the locally nilpotent residual of G.

But $g \in N_G(F_\alpha)$, and so $(\underset{\sim}{S} \cap F_\alpha)^g \leq \underset{\sim}{T} \cap F_\alpha$.

Therefore

$$(\underset{\sim}{S} \cap F_\alpha)^g = \underset{\sim}{T} \cap F_\alpha .$$

Let φ_β be the automorphism of F_β induced by g.

Let

$$A_\lambda = \{ \text{automorphisms of } F_\lambda \text{ induced by conjugation by an element of } R \text{ and which map } \underset{\sim}{S} \cap F_\lambda \text{ onto } \underset{\sim}{T} \cap F_\lambda \}.$$

Then A_λ is finite and non - empty for all $\lambda \in \Lambda$.

If $F_\alpha \leq F_\beta$ define $p_{\beta\alpha} : A_\beta \to A_\alpha$ by

$$p_{\beta\alpha}(\varphi_\beta) = \varphi_\beta |_{F_\alpha} .$$

Clearly the sets $\{ A_\lambda \ ; \ \lambda \in \Lambda \}$ and maps $\{ p_{\beta\alpha} \}$ form a projection set.

Hence, by the usual argument, there exists a locally inner automorphism φ of G which maps $\underset{\sim}{S}$ onto $\underset{\sim}{T}$.

Therefore $\underset{\sim}{S}$ and $\underset{\sim}{T}$ are locally conjugate in G.

$\square$

<u>Lemma 2.1.10</u>. Let H and K be distributive subgroups of a $\bar{\mathfrak{U}}$-group G, and let $\{ R_i \ ; \ i < \omega \}$ be members of the lower locally nilpotent series of G. Suppose that, for all $i < \omega$, there exist α_i an automorphism of G, effected locally by conjugation by elements of the locally nilpotent residual of G, such that $(HR_i)^{\alpha_i} = KR_i$.

Then there exists an automorphism α of G, effected locally by conjugation by elements of the locally nilpotent residual of G, such that

$$(H(\bigcap_{i<\omega} R_i))^{\alpha} = K(\bigcap_{i<\omega} R_i) .$$

<u>Proof</u>. Let $R_\omega = \bigcap_{i<\omega} R_i$. Suppose that there exists an automorphism $\bar{\beta}$ of G/R_ω induced locally by conjugation by elements of R the locally nilpotent residual of G such that

$$(HR_\omega/R_\omega)^{\bar{\beta}} = KR_\omega/R_\omega .$$

Then, by Lemma 2.1.3, there exists an automorphism β of G effected locally by conjugation by elements of R such that

$$(HR_\omega)^{\beta} = KR_\omega .$$

Therefore, without loss of generality, we may assume that $R_\omega = 1$.

Now, $H = \bigcap_{i<\omega} HR_i$ and $K = \bigcap_{i<\omega} KR_i$, and therefore, given $\beta \in \Lambda$ there exists $i < \omega$ such that

$$H \cap F_\beta = HR_i \cap F_\beta \text{ and } K \cap F_\beta = KR_i \cap F_\beta .$$

By hypothesis there exists an element $g \in R$ such that

$$(H \cap F_\beta)^g = (HR_i \cap F_\beta)^g = KR_i \cap F_\beta = K \cap F_\beta .$$

Let

$A_\lambda = \{$ automorphisms of F_λ which are induced by conjugation by elements of R and which map $H \cap F_\lambda$ onto $K \cap F_\lambda \}$.

Then A_λ is finite and non - empty for all $\lambda \in \Lambda$.

If $F_\alpha \leq F_\beta$ define $p_{\beta\alpha} : A_\beta \to A_\alpha$ by

$p_{\beta\alpha}(\varphi_\beta) = \varphi_\beta |_{F_\alpha}$, where φ_β is the automorphism of F_β induced by g, as above.

This is well-defined, since $g \in N_G(F_\alpha)$ and so

$$(H \cap F_\alpha)^g = K \cap F_\alpha .$$

Clearly the sets $\{ A_\lambda \ ; \lambda \in \Lambda \}$ and maps $\{ p_{\beta\alpha} \}$ form a projection set.

Hence, by the usual argument, there exists a locally inner automorphism φ of G effected locally by conjugation by elements of R with the required property.

$\square$

<u>Remark</u>. With the notation as in the preceding lemma, let $h \in H$. Then there exists $F_\lambda \in \Sigma$ such that $h \in F_\lambda$. It is then clear, from the way φ is built up, that

$$h^\varphi = h^{\varphi_\lambda}.$$

We shall define a group $G \in \mathcal{Y}$ if and only if

(i) $G \in \bar{\mathcal{U}}$,

(ii) the Sylow p -complements of N are pronormal in G, for all normal subgroups N of G and for all primes p.

$$\text{Let } \mathcal{V} = \mathcal{Y}^S.$$

The preceding results now readily show that

$$\mathcal{V} < (\mathcal{W}, \text{Linn}) .$$

The particular cases of Theorems 1.5.6 and 1.5.7 now yield us the following result in any QS -closed subclass $\mathcal{K}$ of $\mathcal{V}$:

<u>Theorem 2.1.11</u>. Let G be a $\mathcal{K}$ -group, then G possesses a unique local conjugacy class of $\mathcal{F}$ -projectors.

The following result is the converse to Lemma 1.1.1 (iii)

Lemma 2.1.12. Let S be a Sylow π-subgroup of a $\mathfrak{V}$-group G, and let X be a subgroup of G such that $G = SX$ and $S \cap X = 1$. Then X is a Sylow π'-subgroup of G.

Proof. Suppose, for a contradiction, that X is not a π'-group, then there exists $1 \neq x \in X$ such that x is a π-element.

Then there exists $F \in \Sigma$ such that $x \in F$. Let $x \in T$, a Sylow π-subgroup of F.

By Lemma 2.1.1, $S \cap F$ is a Sylow π-subgroup of F. Therefore, there exists $g \in F$ such that

$$(S \cap F)^g = T.$$

Therefore

$$S^g \cap F = T .$$

Since $g \in G$, $g = ab$, for some $a \in S$ and $b \in X$. Therefore $x \in T = S^b \cap F$. Hence

$$bxb^{-1} \in b(b^{-1}Sb)b^{-1} \cap X = S \cap X = 1,$$

which contradicts the choice of x.

Therefore X is a π'-group.

Now any subgroup of G properly containing X must contain a non - trivial element of S, and so cannot be a π'-group .

Therefore X is a Sylow π'-subgroup of G.

$\square$

The last result of this section generalizes the corresponding result in finite soluble groups $\underset{\sim}{9}$.

<u>Theorem 2.1.13</u>. Let G be a $\bar{\mathfrak{V}}$ -group, and for each prime p let $S_{p'}$ be a Sylow p'-subgroup of G, and let $S_p = \bigcap_{q \neq p} S_{q'}$. Then the set $\underset{\sim}{S} = \{ S_p \}$ forms a Sylow basis of G, and all Sylow bases of G may be obtained in this way.

<u>Proof</u>. Obviously S_p is a p -subgroup of G, for all p. Let $F \in \Sigma$, then, by Lemma 2.1.1, $S_{p'} \cap F$ is a Sylow p'-subgroup of F.

Then,

$$S_p \cap F = \bigcap_{q \neq p} S_{q'} \cap F = \bigcap_{q \neq p} (S_{q'} \cap F)$$

is a Sylow p -subgroup of F, for all $F \in \Sigma$.

Therefore, by Lemma 2.1.1, S_p is a Sylow p -subgroup of G.

Let $x \in < S_p \; ; \; p \in \pi >$, then $x \in \bigcap_{p \in \pi'} S_{p'}$, which is a π-subgroup of G. Therefore x is a π -element.

Hence $< S_p \; ; \; p \in \pi >$ is a π-group.

Therefore $\underset{\sim}{S}$ is a Sylow basis of G.

Now, let $\underset{\sim}{T}$ be a Sylow basis of G with members T_p.

Let

$$T_{p'} = < T_q \; ; \; q \neq p >.$$

Then, by Lemma 1.1.6 (iv) , $T_{p'}$ is a Sylow p'-subgroup of G. Thus,

$$T_p = \bigcap_{q \neq p} T_{q'} \; .$$

□

An example of a $\mathfrak{W}$-group which does not satisfy the above theorem is the following:

for each prime $p \neq 2$, let C_p be a cyclic group of order p and let $C = \mathrm{Dr}_{p \neq 2} \, C_p$. C has an automorphism x of order 2, which maps every element of C to its inverse.

Let $G = CX$ be the semi - direct product of C by $X = \langle x \rangle$.

Clearly X has infinitely many conjugates in G. Choose countably many distinct conjugates of X and index them by the prime numbers different from 2, i.e. $\{ X_p \; ; \; p \neq 2 \}$.

For each $p \neq 2$, let $C_{p'} = \underset{q \neq p}{\mathrm{Dr}}\, C_q$, and let $S_{p'} = C_{p'} X_p$. Then $S_{p'}$ is a Sylow p'-subgroup of G, for all $p \neq 2$.

However, $\bigcap_{p \neq 2} S_{p'} = 1$, which not a Sylow 2 -subgroup of G.

That G is a $\mathcal{W}$-group is clear, since G, as an abelian by finite group, is a $\mathcal{U}$ -group.

2.2 GENERALIZATIONS OF SOME RESULTS DUE TO ALPERIN.

In this section we shall extend some of Alperin's results which appear in his paper on the relationship between Carter subgroups and basis normalizers of finite soluble groups 1.

Unfortunately we have not been able to extend all of them because of our inability (at the moment) to sufficiently restrict locally inner automorphisms of a $\mathfrak{U}$ -group which map one basis normalizer onto another. However ...

Our first result generalizes Alperin's crucial 'extendibility theorem' 1 Theorem B.

Lemma 2.2.1. Let P be a π -subgroup of a $\mathfrak{U}$ -group G, and suppose that P normalizes a Sylow π'-subgroup S of G and that X is a π'-subgroup of G centralized by P. Then there exists a Sylow π'-subgroup T of G normalized by P and containing X.

Proof. Let $F \in \Sigma$, then $P \cap F$ normalizes $S \cap F$ which is a Sylow π-subgroup of F, by Lemma 2.1.1, and $X \cap F$ is a π'-subgroup of F centralized by $P \cap F$.

Therefore, by 1 Theorem B, there exists T_F a Sylow π'-subgroup of F such that

$$T_F \geq X \cap F \text{ and } T_F \text{ is normalized by } P \cap F.$$

Suppose that F_α, and $F_\beta \in \Sigma$ with $F_\alpha \leq F_\beta$.

Then, $X \cap F_\beta \leq T_\beta$ and T_β is normalized by $P \cap F_\beta$.
Therefore $T_\beta \cap F_\alpha$ is a Sylow π'-subgroup of F_α such that $X \cap F_\alpha \leq T_\beta \cap F_\alpha$.

Let $x \in P \cap F_\alpha$, then

$$(T_\beta \cap F_\alpha)^x \leq T_\beta \cap F_\alpha ,$$

$$\text{i.e. } (T_\beta \cap F_\alpha)^x = T_\beta \cap F_\alpha .$$

Therefore $P \cap F_\alpha$ normalizes $T_\beta \cap F_\alpha$.

Let

$$A_\lambda = \{ \text{ Sylow } \pi'\text{-subgroups of } F_\lambda \text{ which contain } X \cap F_\lambda \text{ and are normalized by } P \cap F_\lambda \}.$$

Then A_λ is finite and non - empty for all $\lambda \in \Lambda$.
If $F_\alpha \leq F_\beta$ define $p_{\beta\alpha} : A_\beta \rightarrow A_\alpha$ by

$$p_{\beta\alpha}(T_\beta) = T_\beta \cap F_\alpha .$$

Clearly the sets $\{ A_\lambda \ ; \ \lambda \in \Lambda \}$ and maps $\{ p_{\beta\alpha} \}$ form a projection set.

Hence, by the usual argument, there exists a Sylow π'-subgroup T of G which reduces into each F_λ, such that $T \cap F_\lambda = T_\lambda$, for all $\lambda \in \Lambda$.

Clearly $X \leq T$ and T is normalized by P.

$\square$

Lemma 2.2.2. Let H be a subgroup of a $\mathfrak{U}$-group G, and let φ be a locally inner automorphism of G such that $H \leq H^\varphi$. If φ is effected locally by conjugation by elements of the locally nilpotent residual R of G, then $H = H^\varphi$.

Proof. Suppose, for a contradiction, that $H < H^\varphi$, then $H^{\varphi^{-1}} < H$.

Then, there exists $h \in H$ such that $h \notin H^{\varphi^{-1}}$. So there exists $F \in \Sigma$ such that $h \in F$.

But $(H \cap F)^{\varphi^{-1}} \leq H \cap F$. Therefore

$$h \in H \cap F = (H \cap F)^{\varphi^{-1}} ,$$

which contradicts the choice of h.

Therefore $H = H^{\varphi}$.

$\square$

Theorem 2.2.3. Let H be a subgroup of a $\mathcal{U}$-group G such that H contains a basis normalizer D of G. Suppose that for all primes p, the Sylow p-subgroup of D centralizes some Sylow p-complement of H. Then D is the normalizer of some Sylow basis of G which reduces into H.

Proof. Let $D = N_G(\underset{\sim}{S})$, and let π be the set of primes p such that $S_{p'}$ does not reduce into H.

Let $p \in \pi$ and let D_p and $D_{p'}$ be the Sylow p- and p'-subgroups of D respectively. Let $C = C_H(D_p)$.

By hypothesis, C contains a p-complement of H, and since $D_{p'} \leq C$ it follows that $D_{p'}$ is contained in a p-complement $U_{p'}$ of H centralized by D_p.

By Lemma 2.2.1, there exists a Sylow p-complement $T_{p'}$ of G normalized by D_p and containing $U_{p'}$.

Since $D_{p'} \leq T_{p'}$, D also normalizes $T_{p'}$.

Let $\underset{\sim}{T}$ be the Sylow basis of G associated with the Sylow p-complement system

$$\{ S_{p'} ; p \notin \pi \} \cup \{ T_{p'} ; p \in \pi \}.$$

Now $\underset{\sim}{T}$ reduces into H and clearly $D \leq N_G(\underset{\sim}{T}) = D^*$, say. But there exists a locally inner automorphism φ of G which is effected locally by conjugation by elements of the locally nilpotent residual of G such that $D^* = D^{\varphi}$.

Therefore $D \leq D^* = D^{\varphi}$.

Therefore, by Lemma 2.2.2, $D = D^*$.

$\square$

Corollary 2.2.4. Let H be a locally nilpotent subgroup of a $\mathfrak{U}$-group G, and suppose that H contains a basis normalizer D of G. Then D is the normalizer of some Sylow basis of G which reduces into H.

Our next result is analogous to Theorems 8 and 9 of $\underset{\sim}{1}$.

Theorem 2.2.5. Let H be a subgroup of a $\mathfrak{U}$-group G, let $\{\pi_1, \ldots, \pi_n, \ldots\}$ be a partition of the set of all primes, and suppose that for each i, $1 \leq i < \infty$, a Sylow π_i-subgroup of H normalizes some Sylow basis of G. Then H normalizes some Sylow basis of G.

Proof. Let $F \in \Sigma$, then for each i, $1 \leq i < \infty$, a Sylow π_i-subgroup of $H \cap F$ normalizes some Sylow basis of F.

Therefore, by $\underset{\sim}{8}$ Theorem 3.1.7, $H \cap F$ normalizes some Sylow basis of F.

Let $F_{\alpha} \leq F_{\beta}$ where F_{α} and $F_{\beta} \in \Sigma$. Suppose that $H \cap F_{\beta}$ normalizes the Sylow basis $\underset{\sim}{S}^{\beta}$ of F_{β}.

Then clearly $H \cap F_{\alpha}$ normalizes $\underset{\sim}{S}^{\beta} \cap F_{\alpha}$ which is a Sylow basis of F_{α}.

Let $\Lambda_{\lambda} = \{$ Sylow bases of F_{λ} normalized by $H \cap F_{\lambda}$ $\}$.

Then A_λ is finite and non - empty for all $\lambda \in \Lambda$.
If $F_\alpha \leq F_\beta$ define $p_{\beta\alpha} : A_\beta \to A_\alpha$ by

$$p_{\beta\alpha}(\underset{\sim}{S}^\beta) = \underset{\sim}{S}^\beta \cap F_\alpha .$$

Clearly the sets $\{ A_\lambda \; ; \lambda \in \Lambda \}$ and maps $\{ p_{\beta\alpha} \}$ form a projection set.

Hence, by the usual argument, there exists a Sylow basis $\underset{\sim}{S}$ of G such that $\underset{\sim}{S} \cap F_\lambda = \underset{\sim}{S}^\lambda$, for all $\lambda \in \Lambda$, with $H \cap F_\lambda$ normalizing $\underset{\sim}{S}^\lambda$.

Therefore H normalizes $\underset{\sim}{S}$, as required.

$\square$

<u>Corollary 2.2.6</u>. With the notation employed in the proof of the above theorem, suppose that H_i is the Sylow π_i - subgroup of some basis normalizer of G, $1 \leq i < \infty$. Then H is a basis normalizer of G.

<u>Proof</u>. This is an immediate consequence of Theorem 2.2.5, since, by Lemma 2.2.2, a subgroup of a $\mathfrak{U}$ -group G cannot be locally conjugate to a proper subgroup of itself, when the automorphism involved is effected locally by conjugation by elements of the locally nilpotent residual of G.

$\square$

Our next result generalizes Alperin's Theorem A.

<u>Lemma 2.2.7</u>. Suppose that G is a $\mathfrak{U}$ -group with π-length one. If H is a π -subgroup of G which normalizes the Sylow π'-subgroup S of G and the π'-subgroup K of G, then there exists a Sylow π'-subgroup T of G such that

$K \leqslant T$ and T is normalized by H.

Proof. Let $X = O_{\pi'}(G)$ and $Y = O_{\pi'\pi}(G)$. Then Y/X is the unique Sylow π-subgroup of G/X.

Therefore $\quad HX/X \leqslant Y/X$.

Now, K is a π'-subgroup of G normalized by H, so

$$[H, K] \leqslant Y \cap K \leqslant X.$$

Therefore the π-subgroup HX/X of G/X normalizes the Sylow π'-subgroup S/X of G/X and centralizes the π'-subgroup KX/X.

Therefore, by Lemma 2.2.1, there exists a Sylow π'-subgroup T/X of G/X normalized by HX/X and containing KX/X.

Since X is a π'-group, it follows that T is a Sylow π'-subgroup of G normalized by H and containing K.

$\square$

The following result generalizes Alperin's Theorem 1.

Theorem 2.2.8. Suppose that G is a $\mathcal{V}$-group and that $l_p(G) = 1$, for all primes p. Let D be a basis normalizer of G and let H be a subgroup of G. If $D \leqslant H$ and normalizes the Sylow basis $\underset{\sim}{T}$ of H, then D is the normalizer of some Sylow basis $\underset{\sim}{S}$ of G which reduces into H to $\underset{\sim}{T}$.

Proof. Suppose that $D = N_G(\underset{\sim}{S})$ and $D \leqslant N_H(\underset{\sim}{T})$.
Let $\underset{\sim}{V}$ be a Sylow basis of G which extends $\underset{\sim}{T}$, and let π be the set of primes p such that

$$V_{p'} \neq S_{p'} \,.$$

If $p \notin \pi$, then $T_{p'} \leq V_{p'} = S_{p'}$.

If $p \in \pi$, then the Sylow p-subgroup D_p of D normalizes both $S_{p'}$ and $T_{p'}$. So, by Lemma 2.2.7, there exists a Sylow p'-subgroup $U_{p'}$ of G normalized by D_p, with $T_{p'} \leq U_{p'}$.

Now, $\underset{\sim}{T}$ reduces into $N_H(\underset{\sim}{T})$, and so $\underset{\sim}{T}$ reduces into D. Thus,

$$D_{p'} \leq T_{p'} \leq U_{p'} \text{ , and } D \text{ normalizes } U_{p'} \text{ .}$$

Let $\underset{\sim}{U}$ be the Sylow basis of G associated with the Sylow p-complement system

$$\{ S_{p'} ; p \notin \pi \} \cup \{ U_{p'} ; p \in \pi \} .$$

Then, $\underset{\sim}{U}$ reduces into H such that $\underset{\sim}{U} \cap H = \underset{\sim}{T}$.

Since $D \leq N_G(\underset{\sim}{U}) = D^*$, say, there exists an automorphism φ of G effected locally by conjugation by elements of the locally nilpotent residual of G such that

$$D^* = D^{\varphi}.$$

Therefore, by Lemma 2.2.2, $D = D^*$.

□

Finally we prove a result which generalizes Alperin's Theorem 4.

<u>Theorem 2.2.9</u>. If G is a $\mho_A$-group, then each Carter subgroup of G contains a unique basis normalizer of G.

<u>Proof</u>. In $\mho_A$-groups, the Carter subgroups are abelian. From Corollary 1.6.10 they are locally pronormal, and hence the result is immediate.

□

2.3. FITTING CLASSES OF $\mathfrak{U}$-GROUPS.

In this section we show that the results of Tomkinson 22 on the existence and local conjugacy of $\mathfrak{F}$-injectors, for any Fitting class $\mathfrak{F}$ of FC-groups, can be generalized to suitable subclasses of $\mathfrak{U}$.

A Fitting class of $\mathfrak{U}$-groups is a subclass $\mathfrak{F}$ of $\mathfrak{U}$ such that

(i) if $G \in \mathfrak{F}$ and H ser G, then $H \in \mathfrak{F}$,

(ii) if $\{ S_\lambda ; \lambda \in \Lambda \}$ is a family of serial $\mathfrak{F}$-subgroups of a $\mathfrak{U}$-group G such that $G = \langle S_\lambda ; \lambda \in \Lambda \rangle$, then $G \in \mathfrak{F}$.

Lemma 2.3.1. If $\mathfrak{F}$ is a Fitting class of $\mathfrak{U}$-groups, then $\mathfrak{F} \cap \mathfrak{O}^*$ is a Fitting class of finite soluble groups.

Proof. This is immediate from the definition of $\mathfrak{F}$.

$\square$

Theorem 2.3.2. If $\mathfrak{Y}$ is a Fitting class of finite soluble groups, then $L\mathfrak{Y} \cap \mathfrak{U}$ is a Fitting class of $\mathfrak{U}$-groups.

Proof. (i) Let H ser $G \in L\mathfrak{Y} \cap \mathfrak{U}$, and let

$$h_1, \ldots, h_n \in H.$$

Then there exists a subgroup E of G such that

$$h_1, \ldots, h_n \in E \in \mathfrak{Y}.$$

Now $H \cap E$ ser E which implies that $H \cap E$ is a subnormal subgroup of E.

Therefore $h_1, \ldots, h_n \in H \cap E \in \mathfrak{Y}$, since $\mathfrak{Y}$ is closed under taking subnormal subgroups.

Therefore

$$H \in L\mathfrak{Y} \cap \mathfrak{U}.$$

(ii) Let the $\mathfrak{U}$-group G be generated by $\{ S_\lambda ; \lambda \in \Lambda \}$ a family of serial $L\mathfrak{Y}$-subgroups of G.

Let $g_1, \ldots, g_n \in G$. Then there exists $F \in \Sigma$ such that $g_1, \ldots, g_n \in F$, and there exist $\lambda_1, \ldots, \lambda_r \in \Lambda$ such that

$$g_1, \ldots, g_n \in F \leqslant < S_{\lambda_1}, \ldots, S_{\lambda_r} > = L, \text{ say.}$$

We now need only show that if a $\mathfrak{U}$-group is generated by a finite number of serial $L\mathfrak{Y}$-subgroups then it is itself a $L\mathfrak{Y}$-group.

So, let $H_1, \ldots, H_m$ be serial subgroups of a $\mathfrak{U}$-group G such that $H_1, \ldots, H_m \in L\mathfrak{Y}$ and $G = < H_1, \ldots, H_m >$.

Let $g_1, \ldots, g_n \in G$, then there exists $F \in \Sigma$ such that

$$g_1, \ldots, g_n \in < H_1 \cap F, \ldots, H_m \cap F >$$

and $H_1 \cap F, \ldots, H_m \cap F$ are all serial subgroups of $H_1, \ldots, H_m$ respectively, and so are $\mathfrak{Y}$-groups.

Therefore $< H_1 \cap F, \ldots, H_m \cap F >$ is generated by subnormal $\mathfrak{Y}$-subgroups, and so is itself a $\mathfrak{Y}$-group.

Therefore $G \in L\mathfrak{Y} \cap \mathfrak{U}$, as required.

$\square$

<u>Corollary 2.3.3</u>. The Fitting classes of $\mathfrak{U}$-groups are precisely the $L\mathfrak{Y} \cap \mathfrak{U}$, where $\mathfrak{Y}$ is a Fitting class of finite soluble groups.

Proof. Let G be an $\mathfrak{F}$-group, and let $g_1, \ldots, g_n \in G$. Then there exists $F \in \Sigma$ such that $g_1, \ldots, g_n \in F$.

But F ser G, and so $F \in \mathfrak{F} \cap \mathfrak{O}^*$.

Therefore $G \in L(\mathfrak{F} \cap \mathfrak{O}^*) \cap \mathfrak{U}$.

The converse follows immediately from the previous theorem.

$\square$

Lemma 2.3.4. If $\mathfrak{F}$ is a Fitting class of $\mathfrak{U}$-groups, then $L\mathfrak{F} \cap \mathfrak{U} = \mathfrak{F}$.

Proof. Clearly $\mathfrak{F} \leq L\mathfrak{F} \cap \mathfrak{U}$.

Let $G \in L\mathfrak{F} \cap \mathfrak{U}$. Then G possesses a local system Σ of finite serial $\mathfrak{F}$-subgroups, and $G = \langle F ; F \in \Sigma \rangle$.

Therefore $G \in \mathfrak{F}$. Hence we have the required equality.

$\square$

Let $\mathfrak{X}$ be any class of groups, then an $\mathfrak{X}$-injector of a group G is defined to be an $\mathfrak{X}$-subgroup X of G such that $X \cap S$ is a maximal $\mathfrak{X}$-subgroup of S, for all serial subgroups S of G.

Lemma 2.3.5. Let $\mathfrak{F}$ be a Fitting class of $\mathfrak{U}$-groups. If V is an $\mathfrak{F}$-injector of a $\mathfrak{U}$-group G and H ser G, then $V \cap H$ is an $\mathfrak{F}$-injector of H.

Proof. Now $V \cap H$ ser V, and so $V \cap H$ is an $\mathfrak{F}$-group.

If S ser H, then S ser G, and so $V \cap S$ is a maximal $\mathfrak{F}$-subgroup of S.

But, $V \cap S = (V \cap H) \cap S$, and so $V \cap H$ is an

$\mathfrak{F}$-injector of H.

$\square$

The next result establishes the existence of $\mathfrak{F}$-injectors of a $\mathfrak{U}$-group for any Fitting class $\mathfrak{F}$ of $\mathfrak{U}$-groups. This generalizes 22 Theorem 3.2.

Theorem 2.3.6. Let $\mathfrak{F}$ be a Fitting class of $\mathfrak{U}$-groups. Then, if G is a $\mathfrak{U}$-group, G possesses $\mathfrak{F}$-injectors.

Proof. Let $\Sigma = \{ F_\lambda ; \lambda \in \Lambda \}$ define G.

Since $\mathfrak{F} \cap \mathfrak{S}^*$ is a Fitting class of finite soluble groups, each F_λ possesses $\mathfrak{F}$-injectors, for all $\lambda \in \Lambda$. Let $A_\lambda = \{ \mathfrak{F}$-injectors of $F_\lambda \}$. Then A_λ is finite and non - empty for all $\lambda \in \Lambda$.

If $F_\alpha, F_\beta \in \Sigma$ with $F_\alpha \leq F_\beta$, define $A_\alpha \leq A_\beta$.

Suppose that V_β is an $\mathfrak{F}$-injector of F_β. Then, by Lemma 2.3.5, $V_\beta \cap F_\alpha$ is an $\mathfrak{F}$-injector of F_α.

Define $p_{\beta\alpha} : A_\beta \to A_\alpha$ by $p_{\beta\alpha}(V_\beta) = V_\beta \cap F_\alpha$. Clearly the sets $\{ A_\lambda ; \lambda \in \Lambda \}$ and maps $\{ p_{\beta\alpha} \}$ form a projection set.

Hence there exist $V_\lambda \in A_\lambda$, one for each $\lambda \in \Lambda$, such that if $\alpha < \beta$, then $p_{\beta\alpha}(V_\beta) = V_\alpha$.

Define $V = \bigcup_{\lambda \in \Lambda} V_\lambda$, we shall show that V is an $\mathfrak{F}$-injector of G.

If $x_1, \ldots, x_n \in V$, then there exists $F_\alpha \in \Sigma$ such that $x_1, \ldots, x_n \in F_\alpha$.

Clearly $V \cap F_\alpha = V_\alpha$, so $x_1, \ldots, x_n \in V_\alpha \in \mathfrak{F}$.

Therefore $V \in L\mathfrak{F} \cap \mathfrak{U}$, and so, by Lemma 2.3.4, V is an $\mathfrak{F}$-group.

Now, let S ser G, and let W be an $\mathfrak{F}$-subgroup of S containing $V \cap S$.

For each $\lambda \in \Lambda$, $W \cap F_\lambda$ is an $\mathfrak{F}$-subgroup of $S \cap F_\lambda$ containing $V \cap S \cap F_\lambda = V_\lambda \cap S$.

Since $S \cap F_\lambda$ is a subnormal subgroup of F_λ, $V_\lambda \cap S$ is a maximal $\mathfrak{F}$-subgroup of $S \cap F_\lambda$.

Therefore

$$W \cap F_\lambda = V_\lambda \cap S, \text{ for all } \lambda \in \Lambda.$$

Therefore

$$W = \bigcup_{\lambda \in \Lambda} (W \cap F_\lambda) = \bigcup_{\lambda \in \Lambda} (V_\lambda \cap S) = V \cap S.$$

Therefore $V \cap S$ is a maximal $\mathfrak{F}$-subgroup of S.

Hence V is an $\mathfrak{F}$-injector of G.

$\square$

<u>Theorem 2.3.7</u>. Let $\mathfrak{F}$ be a Fitting class of $\mathfrak{U}$-groups, and let V be an $\mathfrak{F}$-injector of a $\mathfrak{U}$-group G. If $V \leqslant H \leqslant G$, then V is an $\mathfrak{F}$-injector of H.

<u>Proof</u>. Let $\Sigma = \{ F_\lambda ; \lambda \in \Lambda \}$ define G.

By Lemma 2.3.5, $V \cap F_\lambda$ is an $\mathfrak{F}$-injector of F_λ, for all $\lambda \in \Lambda$.

Since $\mathfrak{F} \cap \mathfrak{O}^*$ is a Fitting class of finite soluble groups, and $V \cap F_\lambda \leqslant H \cap F_\lambda \leqslant F_\lambda$, $V \cap F_\lambda$ is an $\mathfrak{F}$-injector of $H \cap F_\lambda$, for all $\lambda \in \Lambda$, by $\underset{\sim}{5}$.

It now follows from the proof of Theorem 2.3.6, that $V = \bigcup_{\lambda \in \Lambda} (V \cap F_\lambda)$ is an $\mathfrak{F}$-injector of $H = \bigcup_{\lambda \in \Lambda} (H \cap F_\lambda)$.

$\square$

Before we prove that any two $\mathfrak{F}$-injectors of a $\mathfrak{U}$-group are locally conjugate, we need to modify the analogous result from the finite theory 5 Satz 1.

Lemma 2.3.8. Let $\mathfrak{F}$ be a Fitting class of finite soluble groups, and suppose that G is a finite soluble group. Then any two $\mathfrak{F}$-injectors of G are conjugate via the nilpotent residual of G.

Proof. Let $|G| = n$, and suppose that the result has been proved for all groups whose order is less than n. The case $n = 1$ being trivial.

Let V_1 and V_2 be any two $\mathfrak{F}$-injectors of G. Let R be the nilpotent residual of G.

Now, by Lemma 2.3.5, $V_1 \cap R$ and $V_2 \cap R$ are $\mathfrak{F}$-injectors of R, and so, by induction, there exists an element y in the nilpotent residual of R such that

$$(V_1 \cap R)^y = V_2 \cap R.$$

Without loss of generality we may assume that $V_1 \cap R = V_2 \cap R = W$, say, and by induction we may assume that W is a normal subgroup of G, since we may suppose that $G = \langle V_1, V_2 \rangle$ and W is a normal subgroup of V_i, for $i = 1,2$.

Let $N_i = N_G(V_i)$, for $i = 1,2$, and let C_i/W be a Carter subgroup of N_i/W, for $i = 1,2$.

Now, $N_i/(N_i \cap R) \cong N_iR/R$ is a nilpotent group, and so

$[V_i, \underbrace{N_i, \ldots, N_i}_{r}] \leq V_i \cap R = W$, for sufficiently large r.

Therefore V_i/W is hypercentral in N_i/W.

Therefore V_i/W is a normal subgroup of C_i/W, for $i = 1,2$.

We shall show that C_i/W is a Carter subgroup of G/W, for $i = 1,2$. Fix i.

Suppose that $C_i^x = C_i$, for some $x \in G$. Then V_i^x is a normal subgroup of C_i.

Therefore

$$V_i V_i^x \in N_0 \mathfrak{F} = \mathfrak{F} .$$

Hence, by the maximality of V_i, $V_i = V_i^x$.

Therefore $x \in N_i$ and so $x \in C_i$.

Therefore C_i/W is a Carter subgroup of G/W, for $i = 1,2$.

Therefore there exists an element r in the nilpotent residual R of G such that

$$C_1^r = C_2.$$

Therefore V_1^r and V_2 are normal subgroups of C_2, and so

$$V_1^r = V_2.$$

$\square$

We are now in a position to show that the $\mathfrak{F}$-injectors of a $\mathfrak{U}$-group form a unique local conjugacy class of subgroups.

<u>Theorem 2.3.9</u>. Let $\mathfrak{F}$ be a Fitting class of $\mathfrak{U}$-groups. Then any two $\mathfrak{F}$-injectors of a $\mathfrak{U}$-group G are locally conjugate in G.

<u>Proof</u>. Let V_1 and V_2 be any two $\mathfrak{F}$-injectors of G, and let $\Sigma = \{ F_\lambda ; \lambda \in \Lambda \}$ define G.

Then, by Lemma 2.3.5, $V_1 \cap F_\lambda$ and $V_2 \cap F_\lambda$ are $\mathfrak{F}$ -injectors of F_λ, for $\lambda \in \Lambda$.

Therefore, by Lemma 2.3.8, there exists an element g in the nilpotent residual of F_λ such that

$$(V_1 \cap F_\lambda)^g = V_2 \cap F_\lambda .$$

Clearly $g \in R$, the locally nilpotent residual of G. Let F_α, $F_\beta \in \Sigma$ with $F_\alpha \leq F_\beta$. Then, by the above, there exists $g_\beta \in R$ such that

$$(V_1 \cap F_\beta)^{g_\beta} = V_2 \cap F_\beta .$$

Therefore

$$(V_1 \cap F_\alpha)^{g_\beta} \leq V_2 \cap F_\alpha ,$$

since $g_\beta \in N_G(F_\alpha)$.

Hence $(V_1 \cap F_\alpha)^{g_\beta} = V_2 \cap F_\alpha$.

Let φ_β be the automorphism of F_β associated with g_β.

Let

$A_\lambda = \{$ automorphisms of F_λ induced by conjugation by elements of R, and which map $V_1 \cap F_\lambda$ onto $V_2 \cap F_\lambda$ $\}$.

Then A_λ is finite and non - empty for all $\lambda \in \Lambda$.

If $F_\alpha \leq F_\beta$ define $p_{\beta\alpha} : A_\beta \to A_\alpha$ by

$$p_{\beta\alpha}(\varphi_\beta) = \varphi_\beta |_{F_\alpha} .$$

Clearly the sets $\{ A_\lambda \ ; \lambda \in \Lambda \}$ and maps $\{ p_{\beta\alpha} \}$ form a projection set.

Hence, by the usual argument, there exists a locally inner automorphism φ of G which maps V_1 onto V_2.

Hence V_1 and V_2 are locally conjugate in G.

□

2.4. NORMAL FITTING CLASSES OF $\mathfrak{V}$-GROUPS.

In this section we shall show that many of the results obtained by Blessenohl and Gaschütz 2, and Lausch 15 concerning normal Fitting classes of finite soluble groups can be extended to the appropriate subclasses of $\mathfrak{V}$.

A Fitting class $\mathfrak{F}$ of $\mathfrak{V}$-groups is called a normal Fitting class, if for all $\mathfrak{V}$-groups G, the $\mathfrak{F}$-injectors of G are normal subgroups of G.

Clearly if $\mathfrak{F}$ is a normal Fitting class of $\mathfrak{V}$-groups then $\mathfrak{F} \cap \mathfrak{S}^*$ is a normal Fitting class of finite soluble groups.

Theorem 2.4.1. If $\mathfrak{G}$ is a normal Fitting class of finite soluble groups, then $L\mathfrak{G} \cap \mathfrak{V}$ is a normal Fitting class of $\mathfrak{V}$-groups.

Proof. By Theorem 2.3.2, $L\mathfrak{G} \cap \mathfrak{V}$ is a Fitting class of $\mathfrak{V}$-groups. Let $\mathfrak{F} = L\mathfrak{G} \cap \mathfrak{V}$.

Let G be a $\mathfrak{V}$-group, and let V be an $\mathfrak{F}$-injector of G.

Let $v \in V$ and $g \in G$, then there exists $F = \Sigma$ such that $v, g \in F$, where Σ is the local system which defines G.

By Lemma 2.3.5, $V \cap F$ is an $\mathfrak{F}$-injector of F, i.e. $V \cap F$ is a $\mathfrak{G}$-injector of F.

Therefore $v^g \in V \cap F$, i.e. V is a normal subgroup of G.

Hence $\mathfrak{F}$ is a normal Fitting class of $\mathfrak{U}$-groups.

$\square$

Corollary 2.4.2. The normal Fitting classes of $\mathfrak{U}$-groups are precisely the $L\mathfrak{G} \cap \mathfrak{U}$, where the $\mathfrak{G}$ run over the normal Fitting classes of finite soluble groups.

The following result extends Cossey's result which can be found in 2 Satz 5.1.

Theorem 2.4.3. Let $\mathfrak{F}$ be a non - trivial normal Fitting class of $\mathfrak{U}$-groups, then the class of locally nilpotent groups is contained in $\mathfrak{F}$.

Proof. Let $G \in L\mathfrak{N} \leq \mathfrak{U}$, then G is generated by finite nilpotent subgroups $\{ F_\lambda ; \lambda \in \Lambda \}$, say.

Now, by 2 Satz 5.1,

$$F_\lambda \in \mathfrak{N}^* \leq \mathfrak{F} \cap \mathfrak{O}^*.$$

Therefore G is generated by a local system of finite $\mathfrak{F}$-subgroups, i.e.

$$G \in L\mathfrak{F} \cap \mathfrak{U} = \mathfrak{F}, \text{ by Lemma 2.3.4.}$$

$$\text{Hence } L\mathfrak{N} \leq \mathfrak{F}.$$

$\square$

Lemma 2.4.4. Let $\mathfrak{G}$ be a Fitting class of finite soluble groups, and let K be a serial subgroup of a $\mathfrak{U}$-group G. Then $K_{L\mathfrak{G}} = K \cap G_{L\mathfrak{G}}$.

Proof. $K \cap G_{L\mathfrak{G}}$ ser $G_{L\mathfrak{G}}$, and so is a normal $L\mathfrak{G}$-subgroup of K, and hence is contained in the $L\mathfrak{Y}$ -radical of K.

Conversely, the $L\mathfrak{Y}$ -radical of K is a serial subgroup of G, and hence is contained in the $L\mathfrak{Y}$ -radical of G, since the latter is generated by all the serial $L\mathfrak{Y}$ -subgroups of G.

Hence we have equality.

$\square$

The following corollary is immediate from the above lemma.

Corollary 2.4.5. Let $\mathfrak{F}$ be a Fitting class of $\mathfrak{U}$ -groups and suppose that the $\mathfrak{U}$ -group G is defined by the local system Σ. Then

$$F \cap G_{\mathfrak{F}} = F_{\mathfrak{F}}, \text{ for all } F \in \Sigma.$$

We are now in a position to extend Satz 5.3 of 2.

Theorem 2.4.6. Let $\mathfrak{F}$ be a non - trivial Fitting class of $\mathfrak{U}$ -groups. Then $\mathfrak{F}$ is a normal Fitting class of $\mathfrak{U}$ -groups if and only if $G' \leqslant G_{\mathfrak{F}}$, for all $G \in \mathfrak{U}$.

Proof. First of all, assume that $\mathfrak{F}$ is a normal Fitting class of $\mathfrak{U}$ -groups.

Let G be a $\mathfrak{U}$ -group, and let $g, h \in G$.
Then there exists $F \in \Sigma$ such that $g, h \in F$, where Σ is the local system that defines G.

Then $[g, h] \in F' \leqslant G'$.
Now, by Corollary 2.4.5, $F \cap G_{\mathfrak{F}} = F_{\mathfrak{F}}$.

Therefore

$F \cap G_{\mathfrak{F}} \geq F'$, by $\underset{\sim}{2}$ Satz 5.3.

Therefore $G' \leqslant G_{\mathfrak{F}}$, as required.

Conversely, suppose that G is a $\mathfrak{V}$-group. Then clearly $G_{\mathfrak{F}} \leqslant V$, for all $\mathfrak{F}$-injectors V of G.

Therefore V is a normal subgroup of G, for all $\mathfrak{F}$-injectors V of G.

Hence $\mathfrak{F}$ is a normal Fitting class of $\mathfrak{V}$-groups.

$\square$

The following corollary of Theorem 2.4.6 extends $\underset{\sim}{2}$ Satz 6.2.

<u>Corollary 2.4.7</u>. Let $\{ \mathfrak{F}_\lambda ; \lambda \in \Lambda \}$ be a family of non - trivial normal Fitting classes of $\mathfrak{V}$-groups. Then

$$\mathfrak{F} = \bigcap_{\lambda \in \Lambda} \mathfrak{F}_\lambda$$

is also a non - trivial normal Fitting class of $\mathfrak{V}$-groups.

<u>Proof</u>. Clearly $\mathfrak{F}$ is a Fitting class of $\mathfrak{V}$-groups.

By Theorem 2.4.3, $L\Pi \leqslant \mathfrak{F}_\lambda$, for all $\lambda \in \Lambda$. Therefore $L\Pi \leqslant \mathfrak{F}$, and hence $\mathfrak{F}$ is non - trivial.

Now, let G be a $\mathfrak{V}$-group, then, by Theorem 2.4.6, $G' \leqslant G_{\mathfrak{F}_\lambda}$, for all $\lambda \in \Lambda$.

Therefore

$$G' \leqslant \bigcap_{\lambda \in \Lambda} G_{\mathfrak{F}_\lambda} \leqslant G_{\mathfrak{F}} .$$

Hence, by Theorem 2.4.6, $\mathfrak{F}$ is normal.

$\square$

We now define the product of two Fitting classes of $\mathfrak{V}$-groups by if $\mathfrak{F}$ and $\mathfrak{Y}$ are Fitting classes, then $G \in \mathfrak{F}\mathfrak{Y} \cap \mathfrak{U}$, if and only if G modulo its $\mathfrak{F}$-radical is a $\mathfrak{Y}$-group.

This notation may be a little confusing with that used in connection with products of arbitrary group classes, where one only requires the existence of a normal $\mathfrak{F}$-subgroup with the above factoring property.

However this is standard notation in the finite theory.

The next corollary extends the analogous result in the finite theory which was first proved by Gaschütz.

Corollary 2.4.8. Let $\mathfrak{F}$ be a Fitting class of $\mathfrak{U}$-groups, and let $\mathfrak{Y}$ be a non - trivial normal Fitting class of $\mathfrak{U}$-groups. Then $\mathfrak{F}\mathfrak{Y} \cap \mathfrak{U}$ is a normal Fitting class of $\mathfrak{U}$-groups.

Proof. (i) Let H ser $G \in \mathfrak{F}\mathfrak{Y} \cap \mathfrak{U}$. Now, $H \cap G_{\mathfrak{F}}$ ser $G_{\mathfrak{F}}$, which is an $\mathfrak{F}$-group, and so $H \cap G_{\mathfrak{F}}$ is also an $\mathfrak{F}$-group.

Clearly $H \cap G_{\mathfrak{F}}$ is the $\mathfrak{F}$-radical of H, since C is generated by all the serial $\mathfrak{F}$-subgroups of G.

Therefore $H \in \mathfrak{F}\mathfrak{Y} \cap \mathfrak{U}$, since

$$H/(H \cap G_{\mathfrak{F}}) \cong HG_{\mathfrak{F}}/G_{\mathfrak{F}} \text{ ser } G/G_{\mathfrak{F}} \in \mathfrak{Y} .$$

(ii) Let

$$G = \langle S_\lambda \text{ ser } G \; ; \; S_\lambda \in \mathfrak{F}\mathfrak{Y} \cap \mathfrak{U} \text{, for all } \lambda \in \Lambda \rangle$$

be a $\mathfrak{U}$-group.

Let K_λ be the $\mathfrak{F}$-radical of S_λ, then $S_\lambda/K_\lambda \in \mathfrak{Y}$, for all $\lambda \in \Lambda$.

Let K be the $\mathfrak{F}$-radical of G, then clearly, since K is generated by all the serial $\mathfrak{F}$-subgroups of G, $S_\lambda \cap K$ equals K_λ, for all $\lambda \in \Lambda$.

Now, $S_\lambda K/K \cong S_\lambda/(S_\lambda \cap K) = S_\lambda/K_\lambda$, which is a $\mathfrak{Y}$-group, for all $\lambda \in \Lambda$.

Hence G/K is generated by a family of serial $\mathfrak{Y}$-subgroups, and so is itself a $\mathfrak{Y}$-group.

Therefore $G \in \mathfrak{F}\mathfrak{Y} \cap \mathfrak{U}$, as required.

Therefore $\mathfrak{F}\mathfrak{Y} \cap \mathfrak{U}$ is a Fitting class of $\mathfrak{U}$-groups.

Let G be a $\mathfrak{U}$-group, then, by Theorem 2.4.6,

$$G' \leqslant G_{\mathfrak{Y}} \leqslant G_{\mathfrak{F}\mathfrak{Y}}.$$

Therefore, by Theorem 2.4.6, $\mathfrak{F}\mathfrak{Y} \cap \mathfrak{U}$ is a normal Fitting class of $\mathfrak{U}$-groups.

$\square$

Our next theorem extends the analogous result of Gaschütz.

<u>Theorem 2.4.9</u>. Let $\mathfrak{F}$ be a non - trivial Fitting class of $\mathfrak{U}$-groups. Then,

(i) $\mathfrak{F}$ is normal if and only if $\mathfrak{F}(L\Pi) = \mathfrak{U}$,

(ii) if $\mathfrak{F} \neq \mathfrak{U}$, then

(a) $(L\Pi)\mathfrak{F} \neq \mathfrak{U}$,

(b) $(L\Pi)^s\mathfrak{F} = (L\Pi)^t\mathfrak{F}$ if and only if $s = t$.

<u>Proof</u>. (i) By Theorem 2.4.6, if G is a $\mathfrak{U}$-group, then $G' \leqslant G_{\mathfrak{F}}$.

Therefore $G \in \mathfrak{F}\mathfrak{A}$.

Conversely, suppose that $\mathfrak{F}(L\Pi) = \mathfrak{U}$, and let

$F \in \Sigma$, where Σ defines G, then

$$F \in (\mathfrak{F} \cap \mathfrak{O}^*)\mathfrak{T}^* = \mathfrak{S}^*.$$

Therefore, by finite case, $\mathfrak{F} \cap \mathfrak{O}^*$ is a normal Fitting class of finite soluble groups.

Hence, by Theorem 2.4.1, $L(\mathfrak{F} \cap \mathfrak{O}^*) \cap \mathfrak{V}$ is a normal Fitting class of $\mathfrak{V}$-groups.

Therefore $\mathfrak{F}$ is a normal Fitting class of $\mathfrak{V}$-groups, since $\mathfrak{F} = L(\mathfrak{F} \cap \mathfrak{O}^*) \cap \mathfrak{V}$.

(ii) (a) Suppose, for a contradiction, that $\mathfrak{S}^* \leqslant \mathfrak{F}$. Then, if G is a $\mathfrak{V}$-group,

$$G \in L\mathfrak{F} \cap \mathfrak{V} = \mathfrak{F}, \text{ by Lemma 2.3.4.}$$

Therefore $\mathfrak{F} = \mathfrak{V}$, which is a contradiction. Therefore $\mathfrak{S}^* \nleqslant \mathfrak{F}$, hence, by the finite case,

$$\mathfrak{T}^*(\mathfrak{F} \cap \mathfrak{O}^*) \neq \mathfrak{S}^*.$$

Therefore

$$(L\mathfrak{T})\mathfrak{F} \neq \mathfrak{V}.$$

(b) Suppose that $(L\mathfrak{T})^s\mathfrak{F} = (L\mathfrak{T})^t\mathfrak{F}$, then

$$(L\mathfrak{T})^s\mathfrak{F} \cap \mathfrak{O}^* = (L\mathfrak{T})^t\mathfrak{F} \cap \mathfrak{O}^*, \text{ i.e.}$$

$$(\mathfrak{T}^*)^s(\mathfrak{F} \cap \mathfrak{O}^*) = (\mathfrak{T}^*)^t(\mathfrak{F} \cap \mathfrak{O}^*).$$

By (a), $\mathfrak{F} \cap \mathfrak{O}^* \neq \mathfrak{S}^*$, so, by the finite case, $s = t$.

$\square$

In the remainder of this section we shall show that Lausch's theorem on normal Fitting pairs in finite soluble groups can be extended to the class of $\mathfrak{V}$-groups.

Let A be an arbitrary abelian group, and let

$$\mathrm{Hom}(\mathfrak{V}, A) = \{ \text{group homomorphisms } \mathfrak{V}\text{-groups to } A \},$$

and let $d : \mathfrak{V} \to \mathrm{Hom}(\mathfrak{V}, A)$ be a mapping.

The pair (A, d) is called a <u>normal Fitting pair</u> if

(N1) $dG : G \to A$.

(N2) whenever G and H are $\mathfrak{V}$-groups, $\alpha : G \to H$ is a monomorphism and αG ser H, then $dG = dH\alpha$,

(N3) $A = \{ dG(g) \; ; \; g \in G \in \mathfrak{V} \}$.

Let $\mathfrak{F}(A, d) = \{ G \in \mathfrak{V} \; ; \; G = \ker(dG) \}$.

Before we prove Lausch's theorem, we need to generalize a result of Gaschütz 2 Satz 3.1.

<u>Theorem 2.4.10</u>. $\mathfrak{F}(A, d)$ is a normal Fitting class of $\mathfrak{V}$-groups, and if R is the $\mathfrak{F}(A, d)$-radical of a $\mathfrak{V}$-group G, then $R = \ker(dG)$, for all $\mathfrak{V}$-groups G.

<u>Proof</u>. Let $\mathfrak{F} = \mathfrak{F}(A, d)$, and let S ser $G \in \mathfrak{F}$.

Then, by (N2), $dS(S) = dG(S) \leq dG(G) = 1$.

Therefore S is an $\mathfrak{F}$-group.

Now, let

$$G = \langle S_\lambda \text{ ser } G \; ; \; S_\lambda \in \mathfrak{F}, \text{ for all } \lambda \in \Lambda \rangle$$

be a $\mathfrak{V}$-group, and let $g \in G$.

Then there exist $\lambda_1, \ldots, \lambda_n \in \Lambda$ such that

$$g = s_{\lambda_1} \ldots s_{\lambda_n}, \text{ for some } s_{\lambda_i} \in S_{\lambda_i}, \; 1 \leq i \leq n,$$

where some of the λ_i's may appear several times if necessary.

Therefore

$$dG(g) = dG(s_{\lambda_1} \ldots s_{\lambda_n}) = dG(s_{\lambda_1}) \ldots dG(s_{\lambda_n})$$

$$= dS_{\lambda_1}(s_{\lambda_1}) \ldots\ldots dS_{\lambda_n}(s_{\lambda_n}) = 1\ldots.1 = 1.$$

Therefore G is an $\mathfrak{F}$-group, and hence $\mathfrak{F}$ is a Fitting class of $\mathfrak{V}$-groups.

Let $K = \ker(dG)$ for a $\mathfrak{V}$-group G. Now, by (N2),

$$dK(K) = dG(K) = 1,$$

and so K is a normal $\mathfrak{F}$-subgroup of G.

Therefore $K \leq G_{\mathfrak{F}}$, the $\mathfrak{F}$-radical of G.

Now, $1 = dG_{\mathfrak{F}}(G_{\mathfrak{F}}) = dG(G_{\mathfrak{F}})$ implies that $G_{\mathfrak{F}} \leq K$.

$$\text{Hence } G_{\mathfrak{F}} = \ker(dG).$$

Now

$$G/G_{\mathfrak{F}} = G/\ker(dG) \cong \operatorname{im}(dG) \leq A.$$

Therefore $G/G_{\mathfrak{F}}$ is an abelian group, and so $G' \leq G_{\mathfrak{F}}$.

Hence, by Theorem 2.4.6, $\mathfrak{F}$ is a normal Fitting class of $\mathfrak{V}$-groups.

$\square$

A normal Fitting class is said to <u>admit a normal Fitting pair (A, d)</u>, if $\mathfrak{F} = \mathfrak{F}(A, d)$, where $\mathfrak{F}$ is the normal Fitting class in question.

Let (A, d) and (A_1, d_1) be normal Fitting pairs. We say that they are <u>isomorphic</u> if there exists an isomorphism $\beta : A \to A_1$ such that $\beta d = d_1$.

<u>Lemma 2.4.11</u>. Let $\mathfrak{F}(A, d)$ and $\mathfrak{F}(A_1, d_1)$ be normal Fitting classes. Then

$\mathfrak{F}(A, d) = \mathfrak{F}(A_1, d_1)$ if and only if (A, d) and (A_1, d_1) are isomorphic.

<u>Proof.</u> Define $d^* : \mathfrak{G}^* \to \mathrm{Hom}(\mathfrak{G}^*, A)$ by $d^*G = dG$,

and $d_1^* : \mathfrak{G}^* \to \mathrm{Hom}(\mathfrak{G}^*, A_1)$ by $d_1^*G = dG$, for all $G \in \mathfrak{G}^*$.

Now,

$$A = \{ d^*F(f) \; ; \; f \in F \in \mathfrak{G}^* \} \text{ and}$$
$$A_1 = \{ d_1^*F(f) \; ; \; f \in F \in \mathfrak{G}^* \}.$$

Clearly (A, d^*) and (A_1, d_1^*) are normal Fitting pairs of finite soluble groups, and

$$\mathfrak{F}(A, d^*) = \mathfrak{F}(A, d) \cap \mathfrak{O}^* = \mathfrak{F}(A_1, d_1) \cap \mathfrak{O}^*$$
$$= \mathfrak{F}(A_1, d_1^*).$$

Therefore, by 15 Propos. 2.1, (A, d^*) and (A_1, d_1^*) are isomorphic, i.e. there exists an isomorphism $\beta : A \to A_1$ such that $\beta d^* = d_1^*$.

Now, let $\Sigma = \{ F_\lambda \; ; \; \lambda \in \Lambda \}$ be a local system which defines a $\mathfrak{U}$-group G.

Let $g \in G$, then there exists $F \in \Sigma$ such that $g \in F$.

Therefore $\beta\, dG(g) = \beta\, dF(g) = \beta\, d^*F(g) = d_1^*F(g)$
$= d_1F(g) = d_1G(g)$, since F ser G.

Therefore $\beta d = d_1$.

Hence the result follows.

$\square$

We now give our generalization of Lausch's theorem.

<u>Theorem 2.4.12.</u> Every non - trivial normal Fitting class of $\mathfrak{U}$-groups admits one and only one normal Fitting pair (upto isomorphic Fitting pairs).

<u>Proof.</u> Let $\mathfrak{F}$ be a non - trivial normal Fitting class

of $\mathfrak{V}$-groups, and let $\mathfrak{Y} = \mathfrak{F} \cap \mathfrak{O}^*$. Then $\mathfrak{Y}$ is a non - trivial normal Fitting class of finite soluble groups.

By 15 Theorem 2.4, there exists a normal Fitting pair (A, d*) such that $\mathfrak{Y} = \mathfrak{F}(A, d^*)$.

Define $d : \mathfrak{V} \to \mathrm{Hom}(\mathfrak{V}, A)$ by the following:

let G be a $\mathfrak{V}$-group defined by the local system Σ, if $g \in G$, then $dG(g) = d^*F(g)$, where $g \in F \in \Sigma$.

(N1) Clearly $dG : G \to A$, for all $G \in \mathfrak{V}$.

(N2) Suppose that G and H are $\mathfrak{V}$-groups, $\alpha : G \to H$ is a monomorphism, and αG ser H.

We must show that $dG = dH\alpha$.

Let Σ and Σ^* be local systems which define G and H respectively. Let $g \in G$.

Then there exists $E \in \Sigma$ such that $g \in E$.

Therefore $\alpha(g) \in \alpha(E) \leqslant F$, for some $F \in \Sigma^*$.

Now, αG ser H, so $\alpha(E)$ is subnormal in F.

Therefore $d^*E = d^*F\alpha$, by the finite case.

Therefore $dG(g) = d^*E(g) = d^*F\alpha(g) = dH\alpha(g)$, for all $g \in G$.

Therefore $dG = dH\alpha$.

(N3) Clearly $A = \{ dG(g) \ ; \ g \in G \in \mathfrak{V} \}$.

Therefore (A, d) is a normal Fitting pair.

We shall show that $\mathfrak{F} = \mathfrak{F}(A, d)$.

Let $G \in \mathfrak{F}(A, d)$, then $G = \ker(dG)$.

Therefore

$$G \in L\mathfrak{Y} \cap \mathfrak{V} \leqslant L\mathfrak{F} \cap \mathfrak{V} = \mathfrak{F}.$$

Conversely, let G be an $\mathfrak{F}$-group, then G is an $L\mathfrak{Y}$-group.

Therefore $G = \ker(dG)$, which implies that $G \in \mathfrak{F}(A, d)$.

Hence $\mathfrak{F} = \mathfrak{F}(A, d)$.

The 'only one' part follows from Lemma 2.4.11.

$\square$

APPENDIX.

<u>A1</u>. There exists a metabelian $\bar{\mathfrak{U}}$ -group of exponent 6 possessing a central 3 -chief factor which is not covered by the normalizer of any Sylow 2 -subgroup, i.e. the normalizer of a Sylow 2 -subgroup is not preserved under epimorphisms. Furthermore, the group does not possess $L\Pi$-projectors.

<u>Proof</u>. Let $H = Dr^{\mathbb{Z}} S_3$, and let $x = (\ldots, a_i, \ldots)$ be an element of $Cr^{\mathbb{Z}} S_3$, where a_i is an element of order 3 in S_3 .

$$\text{Let } G = \langle H, x \rangle = H\langle x \rangle.$$

Then $H = RP$, where $R \cong Dr^{\mathbb{Z}} C_3$ and $P \cong Dr^{\mathbb{Z}} C_2$. Clearly H is an FC -group, and R is the locally nilpotent residual of G.

We need only show that x is contained in a finite subgroup of G which is normalized by R.

In fact R centralizes $\langle x \rangle$, so

$$\langle x \rangle^R = \langle x \rangle .$$

Therefore G possesses a local system Σ of finite subgroups such that $R \leq N_G(\Sigma)$.

$$\text{Hence } G \text{ is a } \bar{\mathfrak{U}} \text{-group.}$$

Now, G/H is a central 3 -chief factor of G, while every Sylow 2 -subgroup of G is in fact a Sylow 2 -subgroup of H and is self - normalizing in G.

Suppose, for a contradiction, that G possesses an $L\Pi$ -projector E, say.

$$\text{Then } G = RE.$$

Let $S = R < x >$, then S is the unique Sylow 3 -subgroup of G.

Therefore $S \cap E$ is the Sylow 3 -subgroup of E.

Let Y be a Sylow 2 -subgroup of E, then $E = (S \cap E)Y$.

Therefore $G = SY$ and $S \cap Y = 1$.

Therefore, by Lemma 2.1.12, Y is a Sylow 2 -subgroup of G.

Therefore $Y = P^{\alpha}$ for some locally inner automorphism α of G.

Without loss of generality we may assume that $P \leqslant E$. Now, if $S \cap E \nleqslant R$, then there exists $r \in R$ such that

$$rx \in S \cap E \quad \text{but} \quad rx \notin R \cap E .$$

Therefore $< P, rx > \leqslant E$, but $< P, rx >$ is not a locally nilpotent group.

Hence E is not an $L\mathfrak{N}$ -projector of G.

$\square$

<u>A2</u>. There exists a ($\mathfrak{W}$, Linn) -group which is not a direct product of a $\mathfrak{U}$ -group and an FC -group; it is metabelian and has trivial centre.

<u>Proof</u>. Let $P \cong C_{p^\infty}$, M a completely reducible P -module over Z_q, where $p \neq q$, and let $G = MP$. The existence of such an M can be found in 1.H.3 of

O. H. KEGEL and B. A. F. WEHRFRITZ, 'Locally Finite Groups' North - Holland Mathematical Library, 1973.

Let H be a subgroup of G, if H contains a C_{p^∞}, then H is a group of the same type as G, otherwise H is a $\mathfrak{U}$ -group. If N is a normal subgroup of G, consider G/N.

Then, if G/N contains a C_{p^∞} it is a group of the same type as G, otherwise G/N is a $\mathcal{U}$-group.

Hence, if we can show that the axioms (X1) - (X4), (A1) - (A4) hold for G, then they will surely hold for any section of G.

Let S be a Sylow p-subgroup of G, then $S \cong C_{p^\infty}$. For,

$$S \cong MS/M \leq MP/M \cong P \cong C_{p^\infty},$$

so, suppose, for a contradiction, that $S \cong C_{p^n}$. Let F be the unique subgroup of P of order p^n.

Then, $MS/M = MF/M$, since G/M has a unique subgroup of order p^n. But MS is a $\mathcal{U}$-group, and S and F are both Sylow p-subgroups of it.

Therefore, there exists $m \in M$ such that

$$S = F^m \leq P^m.$$

But P^m is a Sylow p-subgroup of G, and so $S = P^m \cong C_{p^\infty}$, which is a contradiction.

Therefore $S \cong C_{p^\infty}$, and all Sylow p-subgroups of G complement M in G.

Let R and S be Sylow p-subgroups of G, and let $R = \bigcup_i E_i$, $S = \bigcup_i F_i$, where E_i and F_i are the unique subgroups of order p^i in R and S respectively.

Then, $ME_i = MF_i$ is a $\mathcal{U}$-group and $R \cap ME_i = E_i$, $S \cap MF_i = F_i$, for all $i \geq 0$.

Let $i \leq j$, then there exists $m_j \in M$ such that

$$E_j^{m_j} = F_j.$$

Therefore,

$$E_i^{m_j} = F_i .$$

Let φ_j be the automorphism of ME_j which is induced by conjugation by m_j.

Let

A_i = { automorphisms of ME_i which are induced by conjugation by elements of M and map E_i onto F_i }.

Then A_i is finite and non - empty for all $i \geq 0$.

If $i \leq j$, define $A_i \leq A_j$ and define $p_{ji} : A_j \to A_i$ by

$$p_{ji}(\varphi_j) = \varphi_j|_{ME_i} .$$

Clearly the sets $\{ A_i ; i \geq 0 \}$ and maps $\{ p_{ji} \}$ form a projection set.

Hence, by the usual argument, there exists a locally inner automorphism φ of G effected locally by conjugation by elements of M and which maps R onto S.

Hence the Sylow p-subgroups of G form a unique local conjugacy class of subgroups.

(X1) Let S be a Sylow p -subgroup of a normal subgroup N of G, and let $g \in G$.

Then there exists $m \in M$ such that $S^g = S^m$.

Now, $M = \bigoplus_n M_n$, where M_n is an irreducible P -module over Z_q, for all $n \geq 0$.

Hence there exist $i_1, \ldots, i_n$ such that, if we omit trivial components, $m \in M_{i_1} \oplus \ldots \oplus M_{i_n}$, and so

$$m = (m_{i_1}, \ldots, m_{i_n}), \text{ for some } m_{i_t} \in M_{i_t}, \ 1 \leq t \leq n.$$

Hence we may assume that

$$< S, S^m > \leq (M_{i_1} \oplus \ldots \oplus M_{i_n})P = B, \text{ say.}$$

We shall show that $B \in \mathfrak{U}$.

Since B has a unique Sylow q -subgroup, we need only consider the Sylow p -subgroups.

Without loss of generality we may restrict our attention to the Sylow p -subgroups of B, since any subgroup of B is either a group of the same type as B or a $\mathcal{U}$ -group.

Let R and T be Sylow p -subgroups of B, and let $N = M_{i_1}$. Let $K = \ker($ R-action on $N)$, then N is a faithful irreducible R/K -module over Z_q. Let $K \neq xK \in R/K$, and let $y \in C_N(x)$, $z \in R$.

Then

$$(y^z)^x = y^{zx} = y^{xz} = (y^x)^z = y^z .$$

Therefore $C_N(x)$ is a submodule of N, and hence is either the whole of N or 1.

Hence, by the choice of x, $C_N(x) = 1$.

$$\text{Therefore} \quad R = C_{NR}(x) .$$

Now, suppose that x has order p^n. Let x^* be the unique element of T of order p^n.

$$\text{Then } N < x > = N < x^* > \in \mathcal{U} .$$

Therefore there exists $h \in N$ such that

$$< x >^h = < x^* > .$$

Therefore,

$$R^h = C_{NR}(x)^h = C_{NR}(x^*) = C_{NT}(x^*) = T .$$

Hence the Sylow p -subgroups of NR are conjugate.

By repeating the above argument in each component of $M_{i_1} \oplus \dots \oplus M_{i_n}$, we see that the Sylow p -subgroups of B are conjugate. Hence $B \in \mathcal{U}$.

Therefore, there exists $t \in < S, S^m >$ such that

$$S^t = S^m .$$

(X2) Follows trivially since G is metabelian.

(X3) Let T be any Sylow p -subgroup of G, then

$\underset{\sim}{S} = \{ M, T \}$ is a Sylow basis of G.

(X4) Let $\underset{\sim}{T} = \{ S_q, S_p \}$ be a Sylow basis of a subgroup H of G.

Then there exists S a Sylow p -subgroup of G such that $S_p \leq S$. Then, $\underset{\sim}{S} = \{ M, S \}$ is a Sylow basis of G such that $\underset{\sim}{S} \cap H = \underset{\sim}{T}$.

Now, let $A(G) = \{$ locally inner automorphisms of G which are effected locally by conjugation by elements of M $\}$.

(A1) Obviously $A(G) \leq \mathrm{Naut}(G)$.

(A2) Let H be a subgroup of G, and let $\beta \in A(H)$. without loss of generality we may suppose that there is a Sylow p -subgroup of H contained in P.

If $P \leq H$, then define $\alpha \in A(G)$ by

$$\text{(i) } h^{\alpha} = h^{\beta}, \text{ for all } h \in H,$$

$$\text{(ii) } m^{\alpha} = m, \text{ for all } m \in M.$$

Clearly α agrees with β on H.

If $P \nleq H$, then the Sylow p -subgroups of H are finite, and so β is in fact an inner automorphism of H, which trivially defines an A(G) -automorphism that agrees with β on H.

(A3) Let N be a normal subgroup of G, and let $\beta \in A(G/N)$. Then β is effected locally by conjugation by elements of M.

Let $P = \bigcup_n F_n = \bigcup_n < x_n >$. Then

$$(F_nN/N)^{\beta} = (< x_n > N/N)^{\beta} = < x_n >^{m_n}N/N,$$

for some $m_n \in M$, $n \geq 0$.

(a) $PN/N = 1$, then β is the identity on G/N.

(b) $PN/N \cong C_{p^\infty}$, then $P \cap N \cong C_{p^n}$, for some $n \geq 0$.

Now, let $i > n$, and suppose that

$$x_i^{m_j}N = x_i^{m_k}N \text{ , where } j,k > n.$$

Then,

$$x_i^{m_j m_k^{-1}} \equiv 1 \mod P \cap N \text{ .}$$

But,

$$x_i^{m_j m_k^{-1}} \text{ has order } p^i > p^n \text{ .}$$

Therefore

$$x_i^{m_j} = x_i^{m_k} \text{ .}$$

Let φ_i be the automorphism of MF_i induced by conjugation by m_i .

Let $A_i = \{ \varphi_i \}$, and if $i \leqslant j$ define $A_i \leqslant A_j$ and define $p_{ji} : A_j \to A_i$ by

$$p_{ji}(\varphi_j) = \varphi_i \text{ .}$$

Clearly the sets $\{ A_i \}$ and maps $\{ p_{ji} \}$ form a projection set.

Hence by the usual argument, there exists a locally inner automorphism φ of G effected locally by conjugation by elements of M and which induces β on G/N.

(A4) Does not apply since G is metabelian.

Since the Sylow p -subgroups of G form an A(G) - transitive class of subgroups, the same locally inner automorphisms permute the Sylow bases of G transitively.

$$\text{Hence } G \in (\mathfrak{U}, A) \text{ .}$$

Now, for each natural number n, let Q_n be a faithful irreducible P/C_{p^n} -module over Z_q, and take $M = \bigoplus_n Q_n$.

Form $G = MP$, where P acts on the n-th component as P/C_{p^n} does.

Then G is directly indecomposable and its centre is trivial, as required.

□

I would like to thank Drs. Brian Hartley, Ian Stewart and Stewart Stonehewer for their help in constructing A1 and A2.

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